ALGEBRAIC SURFACES AND FOUR-MANIFOLDS

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1. Structures on manifolds

Definition 1.1. A *topological or C^0-manifold* of dimension n is a Hausdorff topological space X, which admits local charts to $\phi_U\colon U \to \mathbb{R}^n$.

It follows that the transition functions

$$t_{UV} := \phi_V \circ \phi_U^{-1}\colon \phi_U(U \cap V) \to \phi_V(U \cap V)$$

are homeomorphisms.

Definition 1.2. By requiring t_{UV} to respect extra geometric structures on $\mathbb{R}^n$, we produce many other classes of manifolds. In increasing order of restrictiveness, we say X is additionally a

(1) *PL manifold* if t_{UV} are piecewise-linear.
(2) *C^k- or C^∞-manifold* if $t_{UV} \in C^k$ are k-differentiable or smooth.
(3) *real-analytic manifold* if t_{UV} are real-analytic.
(4) *complex-analytic manifold* if t_{UV} is holomorphic: i.e. has a complex-linear differential with respect the identification $\mathbb{R}^n = \mathbb{C}^{n/2}$.

(5) *projective variety* if there is a homeomorphism $X \to V(f_1, \ldots, f_m) \subset \mathbb{CP}^N$ to a smooth vanishing locus of homogenous polynomials.
$$\mathbb{CP}^N := (\mathbb{C}^{N+1} \setminus \{0\})/\sim \text{ with } x \sim \lambda x \text{ for all } \lambda \in \mathbb{C}^*.$$

There is a natural notion of isomorphism for each class of manifold/variety, by declaring that a homeomorphism $f : X \to Y$ defines an isomorphism if it preserves the given structure. This amounts to saying that in local charts, f is continuous, PL, smooth, real-analytic, holomorphic, polynomial, etc.

Warning: In the final case, we should really think of X as having the much weaker Zariski topology, which is generated by complements of vanishing loci of polynomials.

What are the differences between these various structures? Can we classify the set of inequivalent finer structures on a coarser structure? Here are some positive answers, some of which we will discuss in this class.

(1) **Munkres, Morrey, Grauert:** Every C^1-manifold has a unique C^k- and C^∞-structure for all $k \geq 1$. Every smooth manifold admits a unique real-analytic structure.
(2) Every PL manifold in dimension $n \leq 7$ admits a smooth structure. In dimension $n \leq 6$ this smooth structure is unique.
(3) **Kirby-Siebenmann:** There is an obstruction in $H^4(X, \mathbb{Z}_2)$ to a topological manifold admiting a PL structure. Assuming this obstruction vanishes, the PL structures (up to concordance) are a torsor over $H^3(X, \mathbb{Z}_2)$ when $n \geq 5$.
(4) **Moise, Smale:** In dimensions $n = 1, 2, 3$, every topological manifold has a unique smooth structure and in dimensions $n \geq 6$, or $n = 5$, $\partial X = \emptyset$, there are only finitely many (possibly zero) smooth structures on given a topological manifold.
(5) **Kervaire, Milnor, Brieskorn:** There are 28 oriented smooth structures on S^7 which form a cyclic group under the operation of direct sum. They can be constructed as links of the hypersurface singularities in $\mathbb{C}^5$:
$$a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0, \text{ for } k = 1, \ldots, 28.$$
(6) **Stallings, Taubes:** As a C^0-manifold, $\mathbb{R}^n$ has a unique smooth structure for $n \neq 4$ and an uncountable infinity of smooth structures when $n = 4$.
(7) **Teichmüller, Deligne-Mumford:** The complex-analytic structures (up to isomorphism) on a compact oriented surface of genus $g \geq 2$ form a complex orbifold M_g of dimension $3g - 3$.
(8) **Serre, Chow:** If X is a compact complex-analytic manifold, any two variety structures on X (if one exists) are isomorphic.
(9) **Freedman:** If X is a compact oriented simply-connected topological 4-manifold, then $H^2(X, \mathbb{Z})$ admits a symmetric bilinear form Q_X valued in $\mathbb{Z}$. When Q_X is even, it is the unique topological invariant

of X, and when it is odd the two topological types are distinguished by the Kirby-Siebenmann invariant. For instance, the blow-up of a K3 surface at a point and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ are homeomorphic (but not diffeomorphic).

(10) **Rokhlin:** If X admits a smooth structure and $w_2(X) = 0$, then 16 divides the signature of Q_X. So the E_8-*manifold* X with intersection form Q_X of signature -8 admits no smooth structure.

(11) **Donaldson:** If Q_X is positive- or negative-definite, then whenever X admits a smooth structure, $H^2(X,\mathbb{Z})$ admits an orthonormal basis for the quadratic form Q_X.

(12) **Friedman, Morgan:** Let X, Y be algebraic surfaces of Kodaira dimension at least zero and let $f : X \to Y$ be a diffeomorphism. Then f preserves, up to sign, the exceptional curve classes and the canonical class. The plurigenera of X and Y are equal and so the Kodaira dimension is a diffeomorphism invariant.

But there are still many unanswered basic open questions. Some of the simplest are: Does S^6 admit a complex structure? Is there only one smooth structure on S^4? The study of manifolds is vast, and we have to zero in on something. We will focus on the interaction between topological and smooth structures for 4-manifolds, and how gauge theory and complex geometry are useful tools for exploring this interaction. As a warm-up, let's prove that the smooth structures on low-dimensional real space are unique.

Proposition 1.3. *The C^0-manifolds $\mathbb{R}^1$ and $\mathbb{R}^2$ admit unique smooth structures (up to diffeomorphism).*

Sketch. Consider a smooth structure $\widetilde{\mathbb{R}}$ on a topological real line. Then $\widetilde{\mathbb{R}}$ admits an orientation (this is purely topological) and by taking a partition of unity subordinate to coordinate charts, we can produce a smooth vector field F on $\widetilde{\mathbb{R}}$ pointing in the direction of the orientation. Then the flow

$$\phi_F : \mathbb{R} \to \widetilde{\mathbb{R}}$$

satisfying $\phi_F'(t) = F(\phi_F(t))$ is a diffeomorphism from standard $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.

In dimension 2, choose an *almost complex structure* on $S = \widetilde{\mathbb{R}}^2$ i.e. an endomorphism $J \in \mathrm{End}(TS)$ satisfying $J^2 = -I$ which is compatible with the given smooth structure. By the Newlander-Nirenberg theorem in dimension two, J defines a complex structure on S. Then since S is simply connected, it is biholomorphic to either $\mathbb{D}$ or $\mathbb{C}$ by the Riemann mapping theorem, both of which are diffeomorphic to $\mathbb{R}^2$. □

Exercise 1.4. *Let X, Y be vector fields on an almost complex surface (S, J). Show that the* Nijenhuis tensor

$$NN_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

vanishes. Hint: work in a local frame (e, Je) of the tangent bundle.

The condition $NN_J = 0$ (in any dimension) is equivalent to the existence of a complex structure for which J_p is induced by multiplication by i on T_pS for all $p \in S$.

Exercise 1.5. *Can you extend this result to all topological manifolds of dimension 1 and 2 using similar (or different) ideas?*

Remark 1.6. In theory, we should be careful to distinguish when smooth structures on a given topological manifold define the *same* smooth structure versus *diffeomorphic* smooth structures.

For instance, the standard smooth structure on $\mathbb{R}$ and its pullback along the homeomorphism $t \mapsto t^3$ do not lie in a single maximal atlas of smooth charts: Otherwise there would be a transition function of the form $t \mapsto t^{1/3}$ for two charts containing the origin. Rather, $t \mapsto t^3$ defines a diffeomorphism between the two smooth structures.

Before we get to some of the beautiful theorems listed above, we review some of the tools involved in studying the geometry of smooth and complex manifolds.

2. Sheaves, bundles, and connections

Let X be any topological space. A *sheaf of abelian groups* $\mathcal{F}$ is an assignment to each open set $U \subset X$ of an abelian group $\mathcal{F}(U)$ together with the data of restriction maps

$$\begin{aligned} \rho_{UV} \colon \mathcal{F}(U) &\to \mathcal{F}(V) \\ s &\mapsto s\big|_V \end{aligned}$$

for any $V \subset U$, satisfying the following axioms:

(1) $\mathcal{F}(\emptyset) = 0$
(2) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$
(3) $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$
(4) Let $U = \cup_{i \in I} U_i$ be an open cover of an open set. Given *sections* $s_i \in \mathcal{F}(U_i)$ for which $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$ there is a unique $s \in \mathcal{F}(U)$ for which $s\big|_{U_i} = s_i$.

Exercise 2.1. *A* presheaf *on X is defined as above, except that (4) need not hold. Can you give an example of a presheaf on some topological space which is not a sheaf?*

Remark 2.2. We can define sheaves of sets or rings in the same way.

Example 2.3. Let X be a C^0-, PL-, C^∞-, real-analytic, or complex-analytic manifold, or algebraic variety. Then there is a sheaf $\mathcal{O}$ of functions which assigns to a coordinate chart U the continuous, PL, smooth, real-analytic, complex-analytic, or polynomial functions on U, respectively. In various cases, we will give different notations for $\mathcal{O}$ e.g. $\mathcal{C}^0$ for continuous functions, $\mathcal{C}^\infty$ for smooth functions, or $\mathcal{O}^{\mathrm{an}}$ for complex-analytic functions.

Example 2.4. Let A be an abelian group. There is a sheaf $\underline{A}$ of locally constant functions, which assigns $\underline{A}(U) := \{f : U \to A \text{ locally constant}\}$. Common cases are $\underline{\mathbb{Z}}$, $\underline{\mathbb{R}}$, $\underline{\mathbb{C}}$. Note that the constant functions usually fail to be a sheaf because two disjoint open sets can support constant functions taking different values.

Example 2.5. Let $\pi \colon \mathcal{E} \to X$ be a *vector bundle* (in the appropriate category of manifolds). This is a manifold with *trivializations*

$$h_{\mathcal{E}} \colon \pi^{-1}(\mathcal{E}) \to (\mathbb{R}^d \text{ or } \mathbb{C}^d) \times U$$

commuting with the projection to U for which the transition functions $h_V \circ h_U^{-1}$ are fiberwise linear: $U \to GL_d(\mathbb{R} \text{ or } \mathbb{C})$. Then $\mathcal{E}$ defines a sheaf

$$\mathcal{E}(U) := \{\text{sections } s : U \to \pi^{-1}(U)\}.$$

The structure we give $\mathcal{E}$ is as a sheaf of $\mathcal{O}$*-modules*: For each open set U, there is an action of functions $\mathcal{O}(U)$ on $\mathcal{E}(U)$ by fiberwise multiplication, and this action is compatible with restriction.

Definition 2.6. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on a topological space X. A *morphism* $\phi : \mathcal{F} \to \mathcal{G}$ is a group homomorphism $\phi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \subset X$ open, which is compatible with the restriction maps. More generally, when we have sheaves of rings or $\mathcal{O}$-modules, we want $\phi(U)$ to be ring homomorphism or morphism of $\mathcal{O}$-modules.

Exercise 2.7. *Show that* $\ker(\phi)$ *is naturally a sheaf of abelian groups. Find an example where the naively defined* $\operatorname{coker}(\phi)$ *is not a sheaf. How might you rectify this problem? Hint: Describe a procedure which converts any presheaf into a sheaf.*

Exercise 2.8. *Consider the sheaf* $\mathcal{O}$ *of holomorphic functions on* $\mathbb{C}$*. Show that multiplication by* z *induces a map of sheaves from* $\mathcal{O}$ *to itself. What are the kernel and cokernel sheaves of this map?*

Definition 2.9. The *global sections* of a sheaf $\mathcal{F}$ on a topological space X are $H^0(X, \mathcal{F}) := \mathcal{F}(X)$.

For example, $\mathcal{C}^\infty(X) = H^0(X, \mathcal{C}^\infty)$ are the smooth functions on X, $VF(X) = H^0(X, TX)$ are the vector fields on X, and $H^0(X, \mathcal{O}^{\text{an}})$ are the global holomorphic functions on X.

Definition 2.10. Given two vector bundles V, W on X, any linear-algebraic construction will give a new vector bundle, for instance $V \oplus W$, $V \otimes W$, $\mathcal{H}om(V, W)$, V^*, $\text{Sym}^p V$, $\bigwedge^p V$ are all vector bundles.

Warning: Let X be a topological space and let V, W be vector bundles. The sheaf $\mathcal{H}om(V, W)$ assigns to a small open set U the $\mathcal{O}(U)$-module

$$\text{Hom}_{\mathcal{O}(U)}(V(U), W(U))$$

whereas $\text{Hom}(V, W)$ will denote the value $\mathcal{H}om(V, W)(X)$ of this sheaf on the total space.

Example 2.11. Let X be a smooth n-manifold. The sheaf Ω^p of smooth p-forms is defined to be $\Omega^p(U) = \bigwedge^p T^*(U)$ where T^* is the cotangent bundle. More concretely, when U is a coordinate chart to $\mathbb{R}^n$ we have

$$\Omega^p(U) = \left\{\sum f_I(z) dz^I\right\}$$

where $1 \leq i_1 < \cdots < i_p \leq n$ and $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ with $z = (z_i)$ the coordinates on U and f_I smooth functions. There is a map $d\colon \Omega^p \to \Omega^{p+1}$ sending a p-form to a $(p+1)$-form $\omega \mapsto d\omega$.

Warning: When X is a complex manifold, we use the symbol TX (and similarly T^*X, Ω^i, etc) to denote the holomorphic tangent bundle: That is, the tangent bundle of X, with its natural complex, holomorphic structure. If we need to think of TX as a real bundle, we will write $TX_{\mathbb{R}}$.

Warning: The map d is not a map of $\mathcal{O}$-modules: It is NOT the case that for any $f \in \mathcal{C}^\infty(U)$ and any $\omega \in \Omega^p(U)$ that $d(f\omega) = fd\omega$, as would be the case for the definition of a morphism of $\mathcal{O}$-modules. The map d is a morphism of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$. With respect to multiplication by functions, it satisfies the *Liebniz rule*: $d(f\omega) = fd\omega + df \wedge \omega$. This means d is a connection, see below.

Example 2.12. In the case of a complex manifold X, we can define Ω^p as above, instead requiring that $f_I(z)$ is holomorphic. We call these *holomorphic* p-forms. Note that this is different from the sheaf $A^{p,0}$ of *smooth* $(p,0)$-forms which are locally expressions as above, but with $f_I(z)$ smooth functions. More generally, the sheaf $A^{p,q}$ of smooth (p,q)-forms is defined on a coordinate chart to be an expression of the form

$$A^{p,q}(U) := \left\{\sum f_{I,J}(z) dz^I \wedge d\overline{z}^J\right\}$$

with $|I| = p$ and $|J| = q$ and $f_{I,J}$ smooth complex-valued functions. Note that the transition functions of $A^{p,q}$ are not holomorphic for $q \neq 0$.

We now define principal G-bundles and associated bundles.

Definition 2.13. Let G be a Lie group. A *principal G-bundle*

$$\pi\colon P \to X$$

is a space admitting local trivializations $h_U\colon \pi^{-1}(U) \to G \times U$ whose transition functions $h_V \circ h_U^{-1}$ are locally described by a map $t_{UV}\colon U \cap V \to G$ acting on the fibers via left multiplication, $P \times G \to P$.

Note that P admits a (fiber-preserving) right action of G, i.e. a map $P \times G \to P$ because right- and left-multiplication commute. Let $\rho\colon G \to GL(E)$ be a representation on a vector space E. The *associated bundle* of P is the vector bundle $\mathcal{E} \to X$ with trivializing charts to $U \times E$ over the same open sets, with transition functions $\rho \circ t_{UV}$.

From the associated bundle construction, we see there is a natural bijection between principal $GL_r(\mathbb{R})$-bundles and vector bundles of rank r.

Definition 2.14. Let X be a smooth manifold. A *metric* h on a vector bundle $\pi\colon \mathcal{E} \to X$ is a smoothly varying metric on every fiber $h_x(p,q) \to \mathbb{R}$ for $p,q \in \pi^{-1}(x)$. If $\mathcal{E}$ is a complex vector bundle, a *hermitian metric* h is as above, but with $h_x(p,q)$ now a Hermitian metric on every fiber.

A *metric* on X is a metric on the tangent bundle TX, i.e. a smoothly varying metric on the tangent space at any point T_pX.

Exercise 2.15. *Prove that any real vector bundle admits a metric, and any complex vector bundle admits a Hermitian metric. Hint: describe the space of metrics as a fiber bundle over X. What are the fibers? Now use a partition of unity argument.*

Now, we describe connections on bundles:

Definition 2.16. Let $\mathcal{E} \to X$ be a vector bundle. A *connection* ∇ is a map of sheaves of abelian groups $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1$ satisfying the *Liebniz rule*:

$$\nabla(fs) = f\,\nabla(s) + s \otimes df \text{ for all } U,\ f \in \mathcal{O}(U),\ s \in \mathcal{E}(U).$$

Remark 2.17. A connection ∇ induces a map (which, by abuse, we also denote by the same symbol) $\nabla\colon \mathcal{E} \otimes \Omega^p \to \mathcal{E} \otimes \Omega^{p+1}$ we declaring

$$\nabla(s \otimes \omega) = \nabla s \otimes \omega + s \otimes d\omega.$$

Exercise 2.18. *Prove that the* curvature $\nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^2$ *is $\mathcal{O}$-linear, and hence defines a section F_∇ of the vector bundle $\mathcal{H}om(\mathcal{E},\mathcal{E}) \otimes \Omega^2$. We say ∇ is* flat *if $F_\nabla = 0$.*

Example 2.19. The differential d defines a connection on the bundle $\mathcal{O}$. Its curvature $F_d = 0$ because $d^2 f = 0$ for any U and $f \in \mathcal{O}(U)$. Let $\omega \in \Omega^1(X)$ be any 1-form. Then $d + \omega$ is also a connection.

Definition 2.20. Given a vector bundle and flat connection $(\mathcal{E}, \nabla)$, the sheaf of *flat sections* is defined to be

$$L(U) := \{s \in \mathcal{E}(U)\colon \nabla s = 0\}.$$

By the following exercise, the restriction $L\big|_U$ is a constant sheaf for a vector space of dimension $\mathrm{rk}(\mathcal{E})$ over any contractible open set U. We call L a *local system*. It is determined by a representation of the fundamental group.

Exercise 2.21. *Consider a trivializing chart $h_U : \pi^{-1}(U) \to U \times \mathbb{R}^r$ for $\mathcal{E}$ over a contractible open set U and consider the identification of $\mathcal{E}(U)$ with a tuple $(f_1, \dots, f_r)$ of functions $f_i \in \mathcal{O}(U)$. Show that any connection ∇ is of the form*

$$\nabla(f_1, \dots, f_r) = (d + A)(f_1, \dots, f_r)$$

where A is an $r \times r$ matrix of one-forms and d is the coordinate-wise differential. We call A the connection 1-form. *Next prove that $F_\nabla = dA + A \wedge A$ where $A \wedge A$ is matrix multiplication of one-forms via wedge product. Let*

$p \in U$ and consider $v \in \pi^{-1}(p)$. Show that when $F_\nabla = 0$, there is a canonical identification $\pi^{-1}(p) \to L(U)$ gotten by sending

$$v \mapsto s_v(q) := \exp\left(\int_p^q A\right) v.$$

Conclude that we have the following bijections:

(1) *Vector bundles of rank r with a flat connection $(\mathcal{E}, \nabla)$.*
(2) *Local systems on X of rank r.*
(3) *Representations of $\pi_1(X) \to GL_r$ modulo conjugation.*

3. Cohomology

We quickly review cohomology of sheaves, including Cech, de Rham, Dolbeault, and singular cohomology.

Definition 3.1. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X. Let $U_{i_1 \ldots i_p} := U_{i_1} \cap \cdots U_{i_p}$. The *Cech complex* is defined as follows:

$$C^p_{\mathfrak{U}}(X, \mathcal{F}) := \prod_{i_1 < \cdots < i_p} \mathcal{F}(U_{i_1 \ldots i_p})$$

with boundary map $\partial \sigma = \partial^p \sigma$ whose components are given by

$$(\partial \sigma)_{i_0 \ldots i_p} := (-1)^j \sigma_{i_0 \ldots \widehat{i_j} \ldots i_p}\big|_{U_{i_0 \ldots i_p}}.$$

The *Cech cohomology with respect to* $\mathfrak{U}$ is by definition

$$H^p_{\mathfrak{U}}(X, \mathcal{F}) := \ker(\partial_p)/\mathrm{im}(\partial_{p-1})$$

and the *Cech cohomology* $H^p(X, \mathcal{F})$ is the limit over all $\mathfrak{U}$ (partially ordered by refinement of open covers) of the Cech cohomologies with respect to $\mathfrak{U}$.

Exercise 3.2. *Check that $\partial^2 = 0$ and $H^0_{\mathfrak{U}}(X, \mathcal{F}) = H^0(X, \mathcal{F})$ for any $\mathfrak{U}$.*

Example 3.3. Consider the circle S^1 and the constant sheaf $\underline{\mathbb{Z}}$. Take an open cover $S^1 = U_0 \cap U_1$ with U_i intervals. Then the Cech complex is the two-step complex

$$\begin{aligned} \underline{\mathbb{Z}}(U_0) \oplus \underline{\mathbb{Z}}(U_1) &\to \underline{\mathbb{Z}}(U_0 \cap U_1) \\ \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b). \end{aligned}$$

Taking the cohomology, we see $H^p_{\mathfrak{U}}(S^1, \underline{\mathbb{Z}}) = \mathbb{Z}$ for $p = 0, 1$ and is zero for other p. This will be the cohomology for any refinement of the open cover, so $H^p(S^1, \underline{\mathbb{Z}})$ is what we expect.

We now give a couple tools for actually computing Cech cohomology, since the definition (as a limit over open covers) is pretty useless. We say that a sheaf $\mathcal{F}$ on X is *acyclic* if $H^k(X, \mathcal{F}) = 0$ for all $k > 0$.

Theorem 3.4. *Let X be paracompact and Hausdorff and let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of sheaves of abelian groups on X. Then there is a long exact sequence of abelian groups*

$$\begin{aligned} 0 \to & H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to \\ & H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_3) \to \cdots \end{aligned}$$

Sketch. The correct definition of sheaf cohomology is the cohomology of an injective resolution. From this definition and formal properties of injective objects, the theorem follows by the usual diagram chase. We have avoided mention of injective objects for brevity.

So the real content of the statement is that when X is paracompact, Hausdorff, the Cech cohomology computes the cohomology of an injective resolution (sometimes called Grothendieck cohomology). A citation is Godement's book "Theorie des faisceaux," Theoreme 5.10.1. □

Corollary 3.5. *Suppose that $0 \to \mathcal{F} \to I_0 \to I_1 \to \cdots$ is an exact sequence of sheaves, with I_p acyclic. Then*

$$H^p(X, \mathcal{F}) = \frac{\ker(I_p(X) \to I_{p+1}(X))}{\operatorname{im}(I_{p-1}(X) \to I_p(X))}.$$

Proof. We break the sequence up into short exact sequences of sheaves

$$\begin{aligned} 0 \to \mathcal{F} \quad & \to I_0 \to \operatorname{im} \partial_0 \to 0 \\ 0 \to \ker \partial_1 & \to I_1 \to \operatorname{im} \partial_1 \to 0 \qquad \cdots \end{aligned}$$

and note that $\operatorname{im} \partial_i = \ker \partial_{i+1}$, as sheaves, by exactness. Taking the long exact sequences in cohomology of each of these short exact sequences, the acyclicity of I_p implies

$$H^p(\mathcal{F}) = H^{p-1}(\operatorname{im} \partial_0) = H^{p-1}(\ker \partial_1) = H^{p-2}(\operatorname{im} \partial_2) = \cdots = \frac{H^0(\ker \partial_p)}{\partial_{p-1} H^0(I_{p-1})}$$

which is the desired equality. □

Corollary 3.6. *Let $\mathfrak{U}$ be an open cover of X. If $\mathcal{F}$ is acyclic on each $U_{i_1 \dots i_p}$ then $H^p_{\mathfrak{U}}(X, \mathcal{F}) = H^p(X, \mathcal{F})$ computes the Cech cohomology.*

Proof. The complex of *sheaves*

$$0 \to \mathcal{F} \to \textstyle\prod_i \mathcal{F}\big|_{U_i} \to \prod_{i<j} \mathcal{F}\big|_{U_{ij}} \to \cdots$$

where we view $\mathcal{F}\big|_U$ as a sheaf on X by declaring $\mathcal{F}\big|_U(V) := \mathcal{F}(U \cap V)$ is an acyclic resolution of $\mathcal{F}$ and hence its global sections (the Cech complex) computes the cohomology of $\mathcal{F}$. □

Theorem 3.7. *Let X be paracompact and locally contractible. The Cech cohomology $H^p(X, \underline{\mathbb{Z}}) = H^p(X, \mathbb{Z})$ agrees with singular cohomology.*

Exercise 3.8. *We say that a sheaf is* flasque *if the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for any $V \subset U$. Show that a flasque sheaf is acyclic.*

A sheaf $\mathcal{E}$ of smooth or continuous sections of a vector bundle on a manifold is not flasque because smooth and continuous functions can blow up at the boundary of a closed set. But such sheaves are *soft* in that any section over a *closed* set extends to all of X. This slightly weaker property is sufficient to show that $\mathcal{E}$ is acyclic. Note that the sheaf of holomorphic p-forms is *not* in general acyclic on a complex manifold X.

Let X be a smooth manifold. The *de Rham complex* is the resolution of the constant sheaf by acyclic sheaves

$$0 \to \underline{\mathbb{R}} \to \mathcal{C}^\infty = \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots .$$

This sequence of sheaves is exact because on any contractible open set $U \subset X$, every closed p-form ω (i.e. $d\omega = 0$) is exact (i.e. $\omega = d\alpha$). So $H^*(X, \underline{\mathbb{R}})$ is computed by the cohomology of the complex of abelian groups

$$\Omega^0(X) \to \Omega^1(X) \to \cdots$$

which is called *de Rham cohomology.*

Similarly, if X is a complex manifold, there is an exact sequence of sheaves

$$0 \to \Omega^p \to A^{p,0} \to A^{p,1} \to \cdots$$

whose the differential is induced by $\overline{\partial} f = \sum_k \frac{df}{d\overline{z}_k} d\overline{z}_k$ called the *Dolbeault complex.* The exactness at the first joint follows from the fact that a section $\omega \in A^{p,0}(U)$ is holomorphic exactly when $\overline{\partial}\omega = 0$. We conclude that

$$H^{p,q}(X) := H^q(X, \Omega^p) = \frac{\ker(A^{p,q}(X) \to A^{p,q+1}(X))}{\operatorname{im}(A^{p,q-1}(X) \to A^{p,q}(X))},$$

which is called *Dolbeault cohomology.* More generally, given any holomorphic vector bundle $\mathcal{E}$, we the $\overline{\partial}$-complex

$$0 \to \mathcal{E} \to \mathcal{E} \otimes A^{0,0} \to \mathcal{E} \otimes A^{0,1} \to \mathcal{E} \otimes A^{0,2} \to \cdots$$

gives an acyclic resolution, and so the cohomology of the complex $H^0(X, \mathcal{E} \otimes A^{0,\bullet})$ computes $H^p(X, \mathcal{E})$.

Remark 3.9. We can take de Rham (or Dolbeault) cohomology $H^i(X, L)$ with coefficients in a local system L. It is computed as the cohomology of the complex $L \otimes \Omega^\bullet$ (or $L \otimes A^{p,\bullet}$) with the differential acting nontrivially only on the second factor.

Now, let's review properties of singular cohomology. The key ones are:

Theorem 3.10. *Let X be a topological space. The singular cohomology $H^*(X, \mathbb{Z})$ enjoys the following properties:*

(1) *(Functoriality) Given any map of topological spaces $f : X \to Y$ there is a canonical pullback map $f^* : H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$.*
(2) *(Cap product) There is a pairing*

$$H^i(X, \mathbb{Z}) \times H_j(X, \mathbb{Z}) \to H_{j-i}(X, \mathbb{Z})$$
$$(\phi,\, \sigma : \Delta_{0,\dots,j} \to X) \mapsto \phi(\sigma\big|_{\Delta_{0,\dots,i}}) \cdot \sigma\big|_{\Delta_{i,\dots,j}}$$

making homology a module over cohomology.

(3) *(Universal coefficients) There are isomorphisms* $H^i(X,\mathbb{Z})/\mathrm{tors} = H_i(X,\mathbb{Z})^*$ *and* $H^i(X,\mathbb{Z})_{\mathrm{tors}} = H_{i-1}(X,\mathbb{Z})_{\mathrm{tors}}$ *with various pieces of singular cohomology.*

(4) *(Cup product) There is a ring structure* $\cup$ *on cohomology satisfying* $\alpha\cup\beta = (-1)^{pq}\beta\cup\alpha$ *for* $\alpha\in H^p(X,\mathbb{Z})$*,* $\beta\in H^q(X,\mathbb{Z})$*.*

(5) *(Kunneth formula) If* R *is a field, we have a decomposition*

$$H^k(X\times Y,R) = \bigoplus_{i+j=k} H^i(X,R)\otimes H^j(Y,R).$$

(6) *(Poincare duality) If* X *is an oriented topological manifold of dimension* n*, there is a* fundamental class $[X]\in H_n(X,\mathbb{Z})$ *for which the cap product* $\cdot\cap[X]\colon H^i(X,\mathbb{Z})\to H_{n-i}(X,\mathbb{Z})$ *is an isomorphism.*

(7) *(de Rham comparison) If* X *is an oriented smooth manifold of dimension* n*, then singular cohomology* $H^p(X,\mathbb{R})$ *is isomorphic to the de Rham cohomology and the cap product* $H^p(X,\mathbb{R})\otimes H_p(X,\mathbb{R})\to\mathbb{R}$ *is described by integration:* $([\omega],\sigma)\mapsto\int_\sigma\omega$*.*

Notation 3.11. Let $M\subset X$ be a submanifold. We will abuse notation and let $[M]$ denote the fundamental class of M, the pushforward of the fundamental class to $H_{\dim M}(X,\mathbb{Z})$, and even the Poincare dual class in $H^{\operatorname{codim} M}(X,\mathbb{Z})$. Given (7), we can rewrite the operator $\cdot\cup[M]$ on $H^{\dim M}(X,\mathbb{R})$ as the linear functional $\int_M$ on closed forms.

Definition 3.12. A *lattice* $(L,\cdot)$ is a finitely generated, free $\mathbb{Z}$-module together with a symmetric bilinear pairing $\cdot\colon L\otimes L\to\mathbb{Z}$. We say L is *non-degenerate* if there is no nonzero $\ell\in L$ for which $\ell\cdot L=0$. We say L is *unimodular* if the map

$$L\to L^* = \mathrm{Hom}(L,\mathbb{Z})$$
$$\ell\mapsto(x\mapsto\ell\cdot x)$$

is an isomorphism. More generally, the *index* of a non-degenerate lattice is the size $|L^*/L|$ of the *discriminant group* L^*/L. A *Gram matrix* for L is the symmetric integral matrix of inner products $(e_i,\cdot e_j)$ for a basis $\{e_i\}$ of L. The *signature profile* $(\sigma_+,\sigma_-,\sigma_0)$ of a lattice $(L,\cdot)$ is the signature of the induced quadratic form on the vector space $L\otimes_{\mathbb{Z}}\mathbb{R}$.

For a non-degenerate lattice, we usually elide the third entry of the signature profile, writing (σ_+,σ_-). Then, the *signature* of $(L,\cdot)$ is $\sigma_+-\sigma_-$.

Exercise 3.13. *Show that the index* $|L^*/L|$ *is equal to the absolute value of the determinant of any Gram matrix for* L*.*

Exercise 3.14. *Let* M *be an oriented* $4k$*-dimensional topological manifold. Using the above properties of cohomology, prove that* $H^{2k}(M,\mathbb{Z})/\mathrm{tors}$ *has the structure of a unimodular lattice. Similarly, let* M *be an oriented* $4k+2$*-dimensional manifold. Show that* $H^{2k+1}(M,\mathbb{Z})/\mathrm{tors}$ *is a unimodular* symplectic *lattice (i.e.* $\cdot$ *is skew-symmetric).*

We give the following geometric understanding of the cup product:

Proposition 3.15. *Let $M, N \subset X$ be immersed submanifolds of a smooth manifold X, which intersect transversely. Let $[M] \in H^{\operatorname{codim} M}(X, \mathbb{Z})$ and $[N] \in H^{\operatorname{codim} N}(X, \mathbb{Z})$ be the Poincare duals of the pushforwards of the fundamental classes of M and N. Then $[M \cap N] = [M] \cup [N]$ where $M \cap N$ is oriented by comparing the orientations on M, N, X.*

In particular, we will use the following special case:

Corollary 3.16. *Let X be a closed, oriented $2k$-manifold and let M, N be submanifolds of complementary dimension intersecting transversely. Define the sign of an intersection point $\operatorname{sgn}_p(M, N) = \pm 1$ depending on if the concatenation of oriented frames (ν_M, ν_N) is an oriented frame of T_pX. Then*

$$\int_X [M] \cup [N] = \sum_p \operatorname{sgn}_p(M, N).$$

Remark 3.17. If M and N are transverse complex submanifolds of a complex manifold X then $\operatorname{sgn}_p(M, N) = 1$ always: For any $\mathbb{C}$-basis $(e_1, \dots, e_n)$ of T_pX, the $\mathbb{R}$-basis $(e_1, ie_1, \dots, e_n, ie_n)$ is, by definition, oriented. Since we have a complex-linear decomposition $T_pX = T_pM \oplus T_pN$, oriented real bases of T_pM and T_pN of the above form will concatenate to one of the above form and so (ν_M, ν_N) is always compatible with the orientation on X.

4. Characteristic classes

We begin by reviewing the Stiefel-Whitney classes. Let G be a topological group. The *classifying space* BG is a topological space admitting a "universal" principal G-bundle $EG \to BG$ for which homotopy classes of maps $[X, BG]$ are in bijection with principal G-bundles $P \to X$ by pulling back the universal bundle to X.

$$P := X \times_{BG} EG$$

A *characteristic class* of P is by definition the pullback of a cohomology class on BG to X.

Example 4.1. Classifying spaces for linear groups often some kind of Grassmannian. For instance, let $G = GL_1(\mathbb{R}) = \mathbb{R}^*$. Then we can take $EG = \mathbb{R}^\infty \setminus \{0\}$ and $BG = \mathbb{RP}^\infty$ with the map $EG \to BG$ be the projectivization map. This works because EG is contractible and G acts freely on it (which is what we need to produce the classifying space). So real line bundles on X are classified by homotopy classes of maps to $\mathbb{RP}^\infty$.

More generally, the same argument works for $G = GL_d(\mathbb{R})$ by taking $BG = \operatorname{Gr}_d(\mathbb{R}^\infty)$ and EG the frame bundle of the universal d-dimensional vector space over BG. In practice, the ∞ can be thought of as just some fixed number, since for any manifold X of fixed dimension, a homotopy class $c \in [X, BG]$ can be represented as a map into a finite dimensional

Grassmannian. Similarly, for $G = GL_d(\mathbb{C})$, the classifying space is the infinite Grassmannian of complex linear spaces $BG = \mathrm{Gr}_d(\mathbb{C}^\infty)$.

We can also retract onto the classifying spaces BO_d and BU_d because $GL_d(\mathbb{R})$, $GL_d(\mathbb{C})$ deformation-retract onto their maximal compact subgroups O_d and U_d via Gram-Schmidt orthonormalization. Concretely, EG in these cases will be orthogonal frame bundles.

Another way of viewing this retraction is as a *reduction of structure group*: Given any real vector bundle $\mathcal{E} \to X$, it admits a metric h. Then, we can choose trivializations for which the fibers are identified to $\mathbb{R}^d$ with its standard metric. With respect to such trivializations, the transition functions lie in O_d rather than $GL_d(\mathbb{R})$. The same applies for hermitian metrics on complex vector bundles, with the group U_d.

Exercise 4.2. *Show that the limit as d goes to infinity of the cohomology groups $H^*(BU_d, \mathbb{Z})$ stabilizes to an infinitely generated polynomial ring*

$$\mathbb{Z}[c_1, c_2, c_3, \dots]$$

with a generator c_i in each even degree $2i$. Hint: Construct a fibration $BU_{d-1} \to BU_d$. What is the fiber? Now apply the Serre spectral sequence and induction.

Similarly, the cohomology groups $H^*(BO_d, \mathbb{Z}_2)$ stabilize to an infinitely generated polynomial ring

$$\mathbb{Z}_2[w_1, w_2, w_3, \dots].$$

Here we must take coefficients in $\mathbb{Z}_2$ as otherwise, it would be impossible for a cohomology class of degree 1 to have nonzero square. Thus, we can define:

Definition 4.3. The *Chern classes* $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$ of a complex vector bundle $\mathcal{E} \to X$ are the pullbacks of c_i to X along the classifying map.

Definition 4.4. The *Stiefel-Whitney classes* $w_i(\mathcal{E}) \in H^i(X, \mathbb{Z}_2)$ of a real vector bundle $\mathcal{E} \to X$ are the pullbacks of the w_i along the classifying map.

Definition 4.5. The *Pontryagin classes* of a real vector bundle $\mathcal{E}$ are $p_i(\mathcal{E}) := (-1)^i c_{2i}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X, \mathbb{Z})$.

Exercise 4.6. *Prove that $2c_{2i+1}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}) = 0$.*

Note that $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ are $K(\mathbb{Z}_2, 1)$ and $K(\mathbb{Z}, 2)$ spaces respectively. Thus, homotopy classes of maps are given by

$$[X, \mathbb{RP}^\infty] = H^1(X, \mathbb{Z}_2) \text{ and } [X, \mathbb{CP}^\infty] = H^2(X, \mathbb{Z})$$

from which we conclude that real line bundles are classified by $w_1(L) \in H^1(X, \mathbb{Z}_2)$ and complex line bundles are classified by $c_1(L) \in H^2(X, \mathbb{Z})$. We will primarily be interested in the Chern classes.

Proposition 4.7. *Let $L \to X$ be a real or complex line bundle. Let $s \in C^\infty(X, L)$ be a smooth section for which s intersects the zero section of L transversely. Then the vanishing locus $V(s) \subset X$, endowed with an appropriate orientation, is a submanifold representing the Poincare dual of $w_1(L)$ or $c_1(L)$, respectively.*

Proof. The class $w_1 \in H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$ or $c_1 \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$ is represented by the Poincare dual $[H]$ (for any finite dimensional projective space) of a linear hypersurface H. We first homotope the classifying map so that its image lies in a finite dimensional projective space $\mathbb{P}^N$. This is possible because X is a finite CW complex, and by the cellular approximation theorem, any map of CW complexes is homotopic to a cellular map.

A linear hypersurface is the vanishing locus of a section of the universal line bundle on $\mathbb{P}^N$. Assuming H is chosen generically, $c_1(L)$ is represented by the inverse image of H, which is $V(s)$ for a pulled back section s which intersects the zero section transversely. □

An alternative, direct proof that $H^2(X, \mathbb{Z})$ represents smooth complex line bundles uses the *exponential exact sequence*

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 1$$

where the first map is inclusion of locally constant functions into smooth, complex-valued functions, and the second map is exponentiation to smooth non-vanishing functions $\mathcal{O}^*$. A 1-cycle $t \in Z^1_{\mathfrak{U}}(X, \mathcal{O}^*)$ in Cech cohomology is a collection of smooth, non-vanishing functions $t_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ for which $t_{ij}t_{ik}^{-1}t_{jk} = 1$. Consider these modulo 1-boundaries, which are 1-cycles of the form $t_{ij} = t_i t_j^{-1}$ with $t_i \in \mathcal{O}^*(U_i)$.

We can define a line bundle L associated to the 1-cycle t by trivializing over the open set U_i and declaring the transition functions to be t_{ij}. If two 1-cycles are homologous, they differ by a 1-boundary, which corresponds to post-composition the trivializations $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ with multiplication by t_i. Thus, $H^1_{\mathfrak{U}}(X, \mathcal{O}^*)$ is the space of line bundles which admit a trivialization obver $\mathfrak{U}$. Since Cech cohomology is the limit over all open covers $\mathfrak{U}$, and every line bundle (and isomorphism of line bundles) admits a trivialization on some open cover, we conclude that $H^1(X, \mathcal{O}^*)$ is in bijection with complex line bundles modulo isomorphism. This exact argument works also for holomorphic line bundles, taking $\mathcal{O}$, $\mathcal{O}^*$ to be the sheaves of holomorphic, holomorphic non-vanishing functions, respectively.

In the smooth case, $H^1(X, \mathcal{O}) = H^2(X, \mathcal{O}) = 0$ because then $\mathcal{O}$ admits partitions of unity. So the long exact sequence of the exponential exact sequence gives an isomorphism $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$, and one can check directly that the map is the first Chern class c_1.

For real line bundles, consider the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}^* \to \underline{\mathbb{Z}}_2 \to 0$$

where the first map is exponentiation, whose image is the sheaf of positive-valued real functions inside the sheaf $\mathcal{O}^*$ of non-vanishing real functions. As before $H^1(X,\mathcal{O}) = H^2(X,\mathcal{O}) = 0$ and so $H^1(X,\mathcal{O}^*) = H^1(X,\mathbb{Z}_2)$ classifies real line bundles.

Exercise 4.8. *Let X be a smooth closed manifold. Show that $w_1(TX) = 0$ if and only if X admits an orientation.*

Theorem 4.9 (The splitting principle)**.** *Suppose $\mathcal{E} = L_1 \oplus \cdots \oplus L_r$. Then $c(\mathcal{E}) := c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots = \prod(1 + c_1(L_i))$. Furthermore, for any vector bundle $\mathcal{E} \to X$, there is some Y and a map $f\colon Y \to X$ for which $H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is injective, and $f^*\mathcal{E}$ splits as a sum of line bundles.*

Concretely, the theorem implies that any computation involving Chern classes can be performed in terms of the "Chern roots" $x_i := c_1(f^*L_i)$. We will do one such computation carefully, then cease to mention f or Y.

Corollary 4.10. $c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F})$.

Proof. We apply the splitting principle to $\mathcal{E}$ and $\mathcal{F}$, then take the product to find a common Y where both $\mathcal{E}$ and $\mathcal{F}$ split as say $f^*\mathcal{E} = L_1 \oplus \cdots \oplus L_r$ and $f^*\mathcal{F} = M_1 \oplus \cdots \oplus M_s$. Then by functoriality of c under pullback,

$$\begin{aligned} f^*(c(\mathcal{E} \oplus \mathcal{F})) = c(f^*(\mathcal{E} \oplus \mathcal{F})) = c(f^*\mathcal{E} \oplus f^*\mathcal{F}) &\overset{\text{Prop}}{=} c(f^*\mathcal{E})c(f^*\mathcal{F}) \\ &= f^*(c(\mathcal{E}))f^*(c(\mathcal{F})) = f^*(c(\mathcal{E})c(\mathcal{F})). \end{aligned}$$

Finally, we use that f^* is injective to conclude the corollary. □

Exercise 4.11. *Use the splitting principle to show that the top Chern class $c_r(\mathcal{E})$ is represented by the vanishing locus of a generic section of $\mathcal{E}$.*

Exercise 4.12. *Use the splitting principle to compute the Chern classes of $V \otimes W$, $\mathrm{Sym}^2(V)$, $\bigwedge^2 V$ in terms of $c_k(V)$ and $c_k(W)$. Which of these formulas can you generalize to higher powers?*

We have that $e(x_1, \ldots, x_r) = c_r(\mathcal{E})$ where e is the elementary symmetric polynomial. Then, any other symmetric polynomial in the x_i is expressible as some polynomial in $c_r(\mathcal{E})$. For instance, we can define:

Definition 4.13. The *total Chern character* of $\mathcal{E}$ is defined to be $\mathrm{ch}(\mathcal{E}) := \sum e^{x_i}$. In particular $\mathrm{ch}_k(\mathcal{E}) = \sum x_i^k/k!$ are the (rescaled) power symmetric polynomials. Note that $\mathrm{ch}(\mathcal{E}) \in H^*(X,\mathbb{Q})$ is a rational cohomology class, since its definition involves integer division.

Definition 4.14. The *Todd class* of $\mathcal{E}$ is $\mathrm{td}(\mathcal{E}) := \prod \frac{x_i}{1-e^{-x_i}} \in H^*(X,\mathbb{Q})$. The *Todd class* of (a complex manifold) X is $\mathrm{td}(X) := \mathrm{td}(TX)$.

Warning: For a real manifold, one usually considers $\mathrm{td}(TX \otimes \mathbb{C})$, which is not the same as the definition we have given of $\mathrm{td}(X)$.

Exercise 4.15. *Expand the above Taylor series to compute* $\mathrm{ch}(\mathcal{E})$ *and* $\mathrm{td}(\mathcal{E})$ *in terms of the Chern classes* $c_k(\mathcal{E})$*, in cohomological degrees* $0, 2, 4, 6, 8$*.*

Exercise 4.16. *Prove* ch *is a ring homomorphism:* $\mathrm{ch}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{ch}(\mathcal{E}) + (\mathcal{F})$ *and* $\mathrm{ch}(\mathcal{E} \otimes \mathcal{F}) = \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{F})$*.*

5. The Riemann-Roch theorem and its generalizations

We begin by defining the *Euler characteristic* of a sheaf $\mathcal{F}$ to be

$$\chi(\mathcal{F}) = \chi(X, \mathcal{F}) = \textstyle\sum_i (-1)^i h^i(X, \mathcal{F})$$

where $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$. This quantity is defined when $h^i(X, \mathcal{F}) = 0$ for all $i \gg 0$ and is finite for all i. For instance, $\chi(\mathbb{C}, \mathcal{O}^{\mathrm{an}})$ is not well-defined: $h^0(\mathbb{C}, \mathcal{O}^{\mathrm{an}}) = \dim_{\mathbb{C}}\{\text{holomorphic functions on } \mathbb{C}\} = \infty$.

Theorem 5.1. *If X is a compact complex manifold, and $\mathcal{F}$ is a holomorphic vector bundle (or more generally coherent sheaf), then $\chi(X, \mathcal{F})$ is well-defined. In fact $h^i(X, \mathcal{F}) = 0$ for all $i > \dim_{\mathbb{C}} X$.*

Remark 5.2. By the agreement of sheaf and singular cohomology, we have that $h^{2\dim_{\mathbb{C}} X}(X, \underline{\mathbb{C}}) = \mathbb{C}$. But note that $\underline{\mathbb{C}}$ is not a holomorphic vector bundle. It is, rather, a local system.

Theorem 5.3 (Riemann-Roch)**.** *Let C be a compact, connected Riemann surface and let $L \to C$ be a holomorphic line bundle. We have*

$$\chi(L) = \deg(L) + 1 - g$$

where $\deg(L) = \int_C c_1(L)$ *and g is the genus of C.*

Proof. It is useful to introduce the notion of a "point bundle" $\mathcal{O}(p)$. Let $U \ni p$ be a small analytic open set containing p with a chart to the disc $\{z \in \mathbb{C} \colon |z| < 2\}$. We define a holomorphic line bundle with two trivializing charts, U and V. Set $V = \{z \in U \colon |z| \leq 1\}^c \subset C$ and let the transition function $t_{UV}(z) = z^{-1}$. Then $\mathcal{O}(p)$ admits a section $s_p \in H^0(C, \mathcal{O}(p))$ which equals z on the chart U and equals 1 on the chart V.

A *divisor* on C is a $\mathbb{Z}$-linear combination of points $D = \sum n_i p_i$. We can define a line bundle

$$\mathcal{O}(D) := \textstyle\bigotimes_i \mathcal{O}(p_i)^{\otimes n_i}.$$

Note that $\mathcal{O}(D)$ has a meromorphic section $s_D := \prod_i s_{p_i}^{n_i}$ (this section will not be holomorphic if some n_i are negative). We have $\int_C c_1(\mathcal{O}(p)) = 1$ because s_p transversely intersects the zero section and these are both holomorphic curves in the total space of $\mathcal{O}(p)$. Thus, $\int_C c_1(\mathcal{O}(D)) = \sum n_i$ by computation with Chern classes.

For now, suppose that there exists some divisor D for which $L = \mathcal{O}(D)$ (see Proposition 7.1). We can now prove the theorem by induction on $\sum |n_i|$. Consider the exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D + p) \to \mathbb{C}_p \to 0$$

where the lefthand map is given by multiplying a (local) section on an open set U by the restriction of the (global) section $s_p\big|_U \in \mathcal{O}(p)(U)$. The quotient sheaf is the skyscraper sheaf which takes values

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

because multiplication by s_p is invertible away from p, and near p the image is, under a trivialization, the holomorphic functions vanishing at p.

It is easy to see that $h^0(C, \mathbb{C}_p) = \mathbb{C}$ and $h^1(C, \mathbb{C}_p) = 0$ by direct Cech computation for any open cover. From the long exact sequence in cohomology, we conclude that $\chi(\mathcal{O}(D+p)) - \chi(\mathcal{O}(D)) = 1$. The theorem follows by induction, together with the base case $\chi(O) = 1 - g$ which may be taken as the definition of the genus.

If this is unsatisfactory and the reader prefers the topological genus, the key here is to prove Serre duality, which implies $\chi(\mathcal{O}) = -\chi(\Omega^1)$ which in turn shows that $c_1(\Omega^1) = 2g - 2$, then to prove that $c_1(\Omega^1) = 2g_{\text{top}} - 2$ by either the Chern-Gauss-Bonnet formula, the Riemann-Hurwitz formula, or the Poincare-Hopf index theorem.

It remains to show that any line bundle $L = \mathcal{O}(D)$ for some divisor D. This is equivalent to showing that L has a meromorphic section: Let s be a meromorphic section of L, with divisor $\operatorname{div} s = D$ recording the orders of zeroes and poles of s in trivializing charts. Note that $\mathcal{O}(-D)$ has a section s_{-D} whose divisor is $-D$ and so $L \otimes \mathcal{O}(-D)$ has a section $s \otimes s_{-D}$ with no zeroes or poles. Then $s \otimes s_{-D}$ defines a global trivialization of the line bundle $L \otimes \mathcal{O}(-D) = \mathcal{O}$ and so we conclude that $L = \mathcal{O}(D)$. □

We now discuss generalizations of the Riemann-Roch theorem:

Theorem 5.4 (Grothendieck, Hirzebruch)**.** *Let $\mathcal{E} \to X$ be a holomorphic vector bundle on a compact complex manifold. We have*

$$\chi(\mathcal{E}) = \int_X \operatorname{ch}(\mathcal{E})\operatorname{td}(X).$$

Exercise 5.5. *Prove* Noether's formula*: If S is a compact complex surface, and $K = c_1(\Omega^1) \in H^2(S, \mathbb{Z})$ then*

$$\chi(\mathcal{O}_S) = \tfrac{1}{12}(K^2 + \chi_{\text{top}}(S)).$$

Exercise 5.6. *Prove the Riemann-Roch formula for compact complex surfaces S: If $L \to S$ is a holomorphic line bundle,*

$$\chi(L) = \chi(\mathcal{O}_S) + \tfrac{1}{2}(L^2 - L \cdot K).$$

Exercise 5.7. *State and prove a Riemann-Roch formula for holomorphic vector bundles on a Riemann surface.*

Theorem 5.8 (Atiyah-Singer)**.** *Let $\mathcal{E}^\bullet \to X$ be an elliptic complex of smooth complex vector bundles on a compact oriented n-manifold. We have*

$$\chi(\mathcal{E}^\bullet) = (-1)^{n(n-1)/2} \int_X \frac{\text{ch}}{\text{eul}}(\mathcal{E}^\bullet)\text{td}(TX \otimes \mathbb{C})$$

where $\frac{\text{ch}}{\text{eul}}$ is a certain natural way to divide the chern character $\text{ch}(\mathcal{E}^\bullet) := \sum(-1)^i \text{ch}(\mathcal{E}^i)$ by the euler class $\text{eul}(X) \in H^n(X, \mathbb{Z})$.

Warning: In general, the cohomology class $\text{ch}(\mathcal{E})\text{td}(X)$ lies in mixed cohomological degree, and integrating against X only picks out the degree n term. We don't define $\frac{\text{ch}}{\text{eul}}$. [Roughly, I think the Chern character lies in compactly supported cohomology of T^*X so you can integrate against the fibers.] But we only need the index theorem a couple times, and in these cases, it will be relatively clear what to do.

Definition 5.9. An *elliptic complex* $0 \to \mathcal{E}^0 \xrightarrow{d_0} \mathcal{E}^1 \xrightarrow{d_1} \cdots \to \mathcal{E}^n \to 0$ is a complex of smooth vector bundles for which:

(1) The maps d_i are order N *differential operators*: In a trivializing chart U, each coordinate of d_i is of the form

$$\sum_{|I| \leq N} g_I \partial_I$$

where g_I are smooth functions and $\partial_I = \partial_{i_1} \cdots \partial_{i_N}$ is partial differentiation in some number of local coordinates $(x_1, \ldots, x_n)$ on U.

(2) The *symbol complex* of $\mathcal{E}^\bullet$ is exact away from the zero section of $\pi : T^*X \to X$. It is the (now $\mathcal{C}^\infty$-linear) complex of vector bundles

$$0 \to \pi^*\mathcal{E}^0 \xrightarrow{\sigma(d_0)} \cdots \to \pi^*\mathcal{E}^n \to 0$$

where $\sigma(d_i)$ is defined by the replacing $\partial_i \mapsto y_i$ in the top order term $\sum_{|I|=N} g_I \partial_I$. Here $y_i : T^*U \to \mathbb{R}$ is the coordinate function on the second factor of $T^*U = U \times \mathbb{R}^n$ corresponding to x_i.

Exercise 5.10. *Carefully derive that the symbol map well-defined.*

Example 5.11. Consider the differential operator $d \colon \mathcal{O} \to \Omega^1$. In local coordinates on $\mathcal{O}$ and Ω^1, we have that $d = (\partial_1, \ldots, \partial_n)$ is simply coordinate-wise differentiation. Thus, the symbol

$$\sigma(d) : \pi^*\mathcal{O} \to \pi^*\Omega^1$$

sends the function $1 \in \pi^*\mathcal{O}(T^*X) = \mathcal{O}(T^*X)$ to $(y_1, \cdots, y_n) \in \pi^*\Omega^1(X)$. In other words, we are saying that $(y_1, \ldots, y_n)$ is a well-defined global section of $\pi^*\Omega^1$. This is true: Given a point $(p, \alpha) \in T^*X$, a point in the fiber $(\pi^*\Omega^1)_{(p,\alpha)}$ corresponds to a cotangent vector $\beta \in \Omega^1_p X$, and there is an obvious preferred section: $\beta = \alpha$.

More generally, considering the map $d : \Omega^p \to \Omega^{p+1}$, the symbol map $\sigma(d)$ is acts on an element $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ of a frame of Ω^p (and hence also a frame of $\pi^*\Omega^p$) by

$$\sigma(d)(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) = \textstyle\sum_j y_j dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

In other words, we have in coordinates that $\sigma(d) = \sum_j y_j dx^j \wedge -$. To check that the symbol complex is elliptic, it suffices to check exactness over every fiber (p, α) for which $\alpha \neq 0$. This is a complex of vector spaces

$$0 \to \textstyle\bigwedge^0 \mathbb{R}^n \to \bigwedge^1 \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n \to \cdots$$

and the map $\sigma(d)$ is simply $\alpha \wedge -$, so the statement reduces to showing that $\alpha \wedge v = 0$ if and only if $v = \alpha \wedge \beta$ for some γ. Choosing a basis for which $\alpha = e_1$ is a coordinate, this is clear. So the de Rham complex is elliptic.

Suppose X is a manifold of real dimension $2n$. Let $\pm x_i$ be the Chern roots of $TX \otimes \mathbb{C}$. Then $\mathrm{eul}(X) = \prod x_i$. We have by computation:

Exercise 5.12. $\mathrm{ch}(\Omega^\bullet X_\mathbb{R} \otimes \mathbb{C}) = \prod_i (1 - e^{x_i})(1 - e^{-x_i})$

We conclude by the Atiyah-Singer index theorem that

$$\chi_{\mathrm{top}}(X) = (-1)^n \int_X \prod_i \frac{(1 - e^{x_i})(1 - e^{-x_i})}{x_i} \prod_i \frac{x_i}{1 - e^{-x_i}} \frac{-x_i}{1 - e^{x_i}} = \int_X \mathrm{eul}(X)$$

which is called the *Chern-Gauss-Bonnet* theorem.

Example 5.13. Let $\mathcal{E} \to X$ be a holomorphic vector bundle. Consider the acyclic resolution by the $\overline{\partial}$-complex $0 \to \mathcal{E} \otimes A^{0,0} \to \mathcal{E} \otimes A^{0,1} \to \cdots$ whose cohomology of global sections computes the groups $H^i(X, \mathcal{E})$. The symbol $\sigma(\overline{\partial})$ is, by essentially the same computation as above, given at a point $(p, \alpha) \in T^*X$ by the operator $\mathrm{id} \otimes \overline{\alpha} \wedge -$ on the complex of vector spaces $\mathcal{E}_p \otimes \bigwedge^\bullet \mathbb{C}^n$. So the same argument implies that the $\overline{\partial}$-complex is elliptic. We conclude by Atiyah-Singer that

$$\begin{aligned}\chi(\mathcal{E}) = \chi(\mathcal{E} \otimes A^{0,\bullet}) &= (-1)^n \int_X \frac{\mathrm{ch}}{\mathrm{eul}}(\mathcal{E} \otimes A^{0,\bullet}) \mathrm{td}(TX \otimes \mathbb{C}) \\ &= (-1)^n \int_X \mathrm{ch}(\mathcal{E}) \mathrm{ch}(A^{0,\bullet}) \prod_i \frac{1}{x_i} \cdot \frac{x_i}{1 - e^{-x_i}} \cdot \frac{-x_i}{1 - e^{x_i}} = \int_X \mathrm{ch}(\mathcal{E}) \mathrm{td}(X).\end{aligned}$$

[The Chern roots of $A^{0,1}$ are x_i, because we are taking anti-holomorphic 1-forms as opposed to holomorphic vector fields, so there are two sign switches. So $\mathrm{ch}(A^{0,\bullet}) = \prod (1 - e^{x_i})$.] So we recover the Hirzebruch-Riemann-Roch theorem from the Atiyah-Singer index theorem.

6. The Hodge theorems

We will not prove the Atiyah-Singer index theorem, but we will discuss some of the ideas by outlining a proof in a simpler case: Hodge theory.

Let (X, g) be a compact, oriented Riemannian n-manifold, i.e. g is a metric on TX. Then g induces a metric on any tensorial construction involving the tangent bundle, in particular, on the spaces $\Omega^p(X)$ of smooth p-forms on X. By abuse, we will also denote this metric by g. Concretely, if $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_x^*, then $e_{i_1} \wedge \cdots \wedge e_{i_p}$ for $i_1 < \cdots < i_p$ is the orthonormal basis of Ω_x^p. Define a symmetric pairing on $\Omega^p(X)$ by

$$\langle \alpha, \beta \rangle := \int_X g(\alpha, \beta)\, d\nu$$

where $d\nu$ is the volume form induced by g. We introduce:

Definition 6.1. The *Hodge star* operator $* : \Omega^p \to \Omega^{n-p}$ is defined by the property $\alpha \wedge *\beta = g(\alpha, \beta)\, d\nu$. It is an isometry.

Exercise 6.2. *Show that $*^2 = (-1)^{p(n-p)}$.*

Proposition 6.3. *Let $d^* := (-1)^{n(p-1)+1} *d*$. We have $\langle \alpha, d\beta \rangle = \langle d^*\alpha, \beta \rangle$.*

Proof. By Stokes' theorem,

$$\langle \alpha, d\beta \rangle = \int \alpha \wedge *d\beta = \int *\alpha \wedge d\beta = -\int d * \alpha \wedge \beta = \int \beta \wedge *d^*\alpha = \langle \beta, d^*\alpha \rangle.$$

□

This proposition says that d and d^* are *formally adjoint*.

Definition 6.4. The *Laplacian* is the differential operator $\Delta = dd^* + d^*d$ from smooth p-forms to themselves. We say that a p-form is *harmonic* if $\Delta\alpha = 0$. Denote the harmonic p-forms by $\mathcal{H}^p(X)$.

Proposition 6.5. *A smooth p-form α is harmonic if and only if*

$$d\alpha = d^*\alpha = 0.$$

Proof. We have from the Proposition 6.3 that

$$\langle \alpha, \Delta\alpha \rangle = \langle \alpha, dd^*\alpha + d^*d\alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle d^*\alpha, d^*\alpha \rangle.$$

The proposition now follows from the statement that $\langle \beta, \beta \rangle \geq 0$ with equality if and only if $\beta = 0$, since $g(\beta, \beta)$ is pointwise non-negative. □

Proposition 6.6. *The vector subspaces $d\Omega^{p-1}(X)$, $\mathcal{H}^p(X)$, $d^*\Omega^{p+1}(X)$ of $\Omega^p(X)$ are mutually orthogonal.*

Proof. $\langle d\alpha, d^*\beta \rangle = \langle d^2\alpha, \beta \rangle = 0$ so the first and third spaces are othogonal. The middle space is orthogonal to the other two because we can move the d or d^* and take the adjoint. But $d\alpha = d^*\alpha = 0$ for any harmonic form α by Proposition 6.5. □

Observation 6.7. *Suppose that $\Omega^p(X)$ were complete with respect to the L^2-inner product $\langle \alpha, \beta \rangle$.* **Warning:** *it isn't! A decomposition*

$$\Omega^p(X) = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X) \oplus d^*\Omega^{p+1}(X)$$

would then hold: Δ would be a self-adjoint operator on a Hilbert space. By the spectral theorem there would be an orthogonal decomposition $\Omega^p(X) =$

$\bigoplus_\lambda \Omega^p(X)^\lambda$ into eigenspaces, and in particular, we would have a decomposition $\Omega^p(X) = \ker(\Delta) \oplus \operatorname{im}(\Delta)$ into $\lambda = 0$ and $\lambda \neq 0$ pieces. Since $\ker(\Delta) = \mathcal{H}^p(X)$ we would have

$$\operatorname{im}(\Delta) = \mathcal{H}^p(X)^\perp = d\Omega^{p-1}(X) \oplus d^*\Omega^{p+1}(X)$$

with the nontrivial containment $\supset$ following from Proposition 6.6.

Exercise 6.8. *Give an example of an inner product space V (necessarily not complete) and a self-adjoint operator O on it for which $V \neq \ker O \oplus \operatorname{im} O$.*

How do we get around the issue that $\Omega^p(X)$ is incomplete with respect to the inner product? Resolving this issue requires some analysis, at the heart of understanding elliptic differential operators and the index theorem. We will give a rough sketch to give a flavor of the math involved.

Definition 6.9. Let $\mathcal{E} \to X$ be a smooth vector bundle on a compact smooth manifold with a metric g. The *Sobolev space* W_k is the completion of $\mathcal{E}(X)$ with respect to a certain norm. Let $\{f_i\}$ be a smooth partition of unity subordinate to finite trivializing cover $\{U_i\}$ and let $U_i \hookrightarrow \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ be embeddings of each chart into a torus, so that $f_i s$ is identified with a section of the trivial bundle $\mathbb{R}^r \times \mathbb{T}^n \to \mathbb{T}^n$ for any $s \in \mathcal{E}(X)$. The *kth Sobolev norm* on $\mathcal{E}(X)$ can be defined as follows:

$$||s||_k := \textstyle\sum_i ||f_i s||_k \text{ where}$$
$$||f_i s||_k := \sum_{|I| \leq k} ||\partial_I(f_i s)||_{L^2}.$$

This definition produces a space W_k which does not depend on the partition of unity or smooth trivializations. Furthermore, W_0 is the L^2-completion of $\mathcal{E}(X)$ (with respect to any metric) on $\mathcal{E}$. Heuristically, the completion describes sections for which the L^2-norms of s and all its partial derivatives up to kth order are finite. This definition is well-suited to studying differential operators, since an order ℓ differential operator from $\mathcal{E} \to \mathcal{F}$ gives a map of Sobolev spaces $W_k(\mathcal{E}) \to W_{k-\ell}(\mathcal{F})$. So we can use tools from the theory of linear maps of Banach spaces. [**Warning:** To actually show that this gives a *bounded* operator between Banach spaces is a little delicate, requiring some simple elliptic estimates.]

The reason for embedding into the torus $\mathbb{T}^n$ instead of $\mathbb{R}^n$ is that it is easiest to analyze Sobolev spaces using Fourier series:

Exercise 6.10. *Consider the trivial bundle on $\mathbb{T}^n$. Then*

$$W_k = \left\{ \textstyle\sum_{\psi \in \mathbb{Z}^n} c_\psi \, e^{2\pi i \psi \cdot \vec{x}} \,\middle|\, \sum_{\psi \in \mathbb{Z}^n} (1 + |\psi|^2 + \cdots + |\psi|^{2k}) c_\psi < \infty \right\}.$$

Theorem 6.11 (Sobolev Embedding Theorem)**.** *There is an embedding $W_n \hookrightarrow \mathcal{C}^0\mathcal{E}(X)$ into continuous sections of $\mathcal{E}$. More generally, there is an embedding $W_{n+k} \hookrightarrow \mathcal{C}^k\mathcal{E}(X)$ and hence $\cap_{m \geq 0} W_m = \mathcal{E}(X)$.*

Sketch. The Fourier coefficients of each coordinates of $f_i s$ must satisfy the growth condition in Exercise 6.10. This implies convergence of $f_i s$ to a continuous function on $\mathbb{T}^n$: We have $c_\psi \in \mathcal{O}(|\psi|^{-2n})$ and $\sum_{\psi\in\mathbb{Z}^n\setminus\{0\}} |\psi|^{-2n}$ converges. The statement about higher derivatives follows by noting that the derivative of anything in W_m lies in W_{m-1}. □

Exercise 6.12. *The embedding $W_k \hookrightarrow W_{k'}$ for any $k > k'$ is a* compact operator*: It sends the unit ball to a compact set.*

Theorem 6.13 (Elliptic regularity and bootstrapping)**.** *Let*

$$0 \to \mathcal{E} \xrightarrow{D} \mathcal{F} \to 0$$

be an elliptic differential operator D of order ℓ. Then $D : W_k(\mathcal{E}) \to W_{k-\ell}(\mathcal{F})$ is Fredholm*: It has finite-dimensional kernel, cokernel, and closed image. Furthermore, the kernel and cokernel of D are isomorphic for all $k \in \mathbb{Z}$, and so are represented by smooth sections.*

Sketch. The idea is to construct a so-called *pseudo-inverse* P to D: It is an operator for which $I-PD$ and $I-DP$ are compact. It is roughly constructed as follows: P inverts D *up to leading differential order* by dividing the Fourier coefficient by the polynomial encoding the symbol. This is possible exactly because the symbol $\sigma(D)$ is invertible.

Example 6.14. Consider the operator $d \colon \mathcal{O} \to \Omega^1$ on the circle S^1 (whose symbol is y). This acts on Fourier series by

$$D : \textstyle\sum_{a\in\mathbb{Z}} c_a e^{2\pi i a x} \mapsto \sum_{a\in\mathbb{Z}} a c_a e^{2\pi i a x}.$$

To construct a pseudo-inverse to D, we define

$$P : \textstyle\sum_{a\in\mathbb{Z}} c_a e^{2\pi i a x} \mapsto \sum_{a\in\mathbb{Z}\setminus\{0\}} a^{-1} c_a e^{2\pi i a x}.$$

Then we have that

$$I - PD : f(x) \mapsto f(0) = a_0$$
$$I - DP : f(x) \mapsto f(0) = a_0$$

both have 1-dimensional cokernel. The Laplacian Δ has symbol $y_1^2 + \cdots + y_n^2$ (or more canonically, the metric $g \in \mathrm{Sym}^2 T^*X$). So the pseudoinverse P of Δ will send the Fourier coefficients $c_\psi \mapsto |\psi|^{-2} c_\psi$ for non-zero ψ and $c_{\vec{0}} \mapsto 0$.

We conclude that $I - PD$ (and similarly $I - DP$) is *regularity increasing*: They map W_k into $W_{k'}$ for some $k' > k$ because the leading order derivative is cancelled off. But by the previous exercise, this embedding is a compact operator. The partition of unity $\{f_i\}$ will interfere with the argument a bit, but any time a derivative in the differential operator interacts with a smooth function f_i instead of the section s, the regularity of s will fail to decrease and the regularity of f_i will stay the same (since it is smooth). So these issues can be swept into the compact operator.

It is easy to show that $I - K$ is a Fredholm operator, in fact of index $= \dim(\ker) - \dim(\mathrm{coker}) = 0$, for any compact operator K. This follows from a continuity argument for the family of operators $I - tK$. It follows that PD and DP are Fredholm, from which it follows that D and P are too.

It remains to show that the kernel and cokernel of D don't depend on the choice of k. If, for instance, $s \in \ker(D)$, then $s = (I - PD)s$ and hence $s \in W_k \implies s \in W_{k'}$ for $k' > k$. Therefore, $s \in \bigcap_m W_m$ and so by Sobolev embedding, the kernels of D and P are represented by smooth sections. □

Remark 6.15. Any elliptic complex $(\mathcal{E}^\bullet, \partial)$ can be turned into a two-term elliptic complex $\mathcal{E}^{\mathrm{even}} \xrightarrow{D} \mathcal{E}^{\mathrm{odd}}$ by taking $D = \sum_i \partial_{2i} + \partial^*_{2i+1}$.

Theorem 6.16 (Hodge Theorem)**.** *Let (X, g) be a compact oriented Riemannian manifold. There is a decomposition*

$$\Omega^p(X) = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X) \oplus d^*\Omega^{p+1}(X)$$

and the map $\mathcal{H}^p(X) \to H^p(X) = \ker(d)/\mathrm{im}(d)$ is an isomorphism (i.e. the harmonic forms are canonical representatives of cohomology classes).

Proof. We have seen that the stated decomposition holds by the Sobolev space argument, and Observation 6.7. We claim that $\ker(d) = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X)$. Certainly $\supset$ is clear. Supposing that $dd^*\beta = 0$, we have $\langle dd^*\beta, \beta\rangle = \langle d^*b, d^*b\rangle = 0$ and so $d^*\beta = 0$, implying the other inclusion $\subset$. Thus, we see that $\ker(d)/\mathrm{im}(d) = H^p(X) \cong \mathcal{H}^p(X)$. □

We have until now been discussing compact oriented real manifolds. What extra structure do we see when X is complex?

Definition 6.17. Let X be a complex manifold. A *Kahler form* $\omega \in \Omega^2(X_{\mathbb{R}})$ is a closed, real 2-form for which $\omega(v, Jw) = g(v, w)$ is a metric on the tangent space $v, w \in T_x X_{\mathbb{R}}$. The *hermitian metric* is $h := g + i\omega$. It defines a positive-definite Hermitian form on the tangent space $T_x X$ (viewed as a complex vector space).

Remark 6.18. Any two of the three structures g, J, ω determine the third structure uniquely. Though notably, given ω not all J work, because the positive-definiteness of g is not automatic. We say that J is *tamed by ω* when $\omega(v, Jw)$ is a positive-definite metric.

Exercise 6.19. *Show that the space of ω-tame almost complex structures J on (X, ω) is connected.*

In some cases, invariants are independent under deformations of J, thus giving invariants of symplectic manifolds (X, ω).

The Hodge theorem works just as well in the Kahler setting, using adjoints with respect to the Hermitian metric h instead of real metric g, but we have some extra structure. Note that there is a decomposition of k-forms

$$\Omega^k(X_{\mathbb{R}}) = \bigoplus_{p+q=k} A^{p,q}(X)$$

into forms of (p,q)-type. But, $d = \partial + \overline{\partial}$ acts by sending a (p,q)-form to the sum of a $(p+1,q)$-form and a $(p,q+1)$-form, so d does not act well with respect to the bigrading (p,q). To rectify this, we have:

Proposition 6.20 (The Kahler identities)**.** *Let (X,h) be a Kahler complex manifold. Let $\Delta = \Delta_d$ be the metric Laplacian for the Hermitian metric h, as above. Define*

$$\Delta_{\overline{\partial}} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$
$$\Delta_{\partial} := \partial\partial^* + \partial^*\partial$$

by taking the formal adjoints to $\overline{\partial}$ and $\partial = d - \overline{\partial}$. Then

$$\tfrac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\overline{\partial}}.$$

Proof. See Griffiths and Harris, or the internet. Even though the computation is purely local, these are surprisingly tricky to prove. □

The upshot of the Kahler identities is that $\ker(\Delta_d) = \ker(\Delta_{\overline{\partial}})$ but $\overline{\partial}, \overline{\partial}^*$ respect the bigrading, sending $A^{p,q}(X) \to A^{p,q\pm 1}(X)$. We conclude:

Theorem 6.21 (Hodge decomposition and symemtry)**.** *Let (X,h) be a Kahler complex manifold. We have a decomposition*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

into harmonic forms of (p,q)-type. Furthermore, $\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}$ and the map $\mathcal{H}^{p,q}(X) \to H^q(X,\Omega^p)$ (via Dolbeault cohomology, see Section 3) is an isomorphism.

Proof. Since $\Delta_d = 2\Delta_{\overline{\partial}}$, the d-harmonic forms are the same as $\overline{\partial}$-harmonic forms, implying that $\ker(\Delta_{\overline{\partial}}) = \ker(\Delta_d\big|_{A^{p,q}})$. By the above arguments, there is an orthogonal decomposition

$$A^{p,q}(X) = \overline{\partial}A^{p,q-1}(X) \oplus \ker(\Delta_{\overline{\partial}}) \oplus \overline{\partial}^* A^{p,q+1}(X)$$

and so $\mathcal{H}^{p,q}(X)$ represents Dolbeault cohomology. □

This wonderful theorem has many interesting implications, for instance:

Exercise 6.22. *Show that if X admits a Kahler structure, $b_{2k+1}(X)$ is even. Consider for example $S := \widetilde{S}/\mathbb{Z}$ where $\widetilde{S} := \mathbb{C}^2 \setminus \{0\}$ and a generator of $\mathbb{Z}$ acts by $(x,y) \mapsto (2x,2y)$. Prove that S admits no Kahler structure!*

Remark 6.23. Every smooth projective variety is Kahler, because $\mathbb{CP}^n$ admits a Kahler form (whose underlying ω is the Fubini-Study form). Its restriction to any smooth projective subvariety of $\mathbb{CP}^n$ is Kahler.

Theorem 6.24 (Hard Lefschetz Theorem)**.** *Define operators $E := [\omega] \wedge -$, its adjoint $F := E^*$, and $H = [E,F]$. Then E, F, H define a representation of $sl_2(\mathbb{R})$ on the harmonic forms $\mathcal{H}^\bullet(X)$ with weight spaces $H\big|_{\mathcal{H}^k(X)} = k-n$. In particular, $E^k : \mathcal{H}^{n-k}(X) \to \mathcal{H}^{n+k}(X)$ is an isomorphism.*

Furthermore, ω is harmonic of $(1,1)$-type and so E sends harmonic (p,q)-forms to harmonic $(p+1,q+1)$-forms.

Corollary 6.25. $h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X) = h^{n-q,n-p}(X)$.

Corollary 6.26. *If X is Kahler, then $h^{p,p}(X) > 0$ for all X, and in particular $b_{2k}(X) > 0$, for $0 \leq k \leq n$.*

7. Introduction to algebraic surfaces

We now have most of the necessary tools at our disposal to classify smooth, projective complex surfaces. We follow Beauville's book. The classification is attributed to Enriques and Kodaira. Let's set some notation:

S is a smooth projective complex surface.

$K = K_S = \Omega^2$ is the *canonical bundle*. Generally $K_X = \Omega^{\dim_{\mathbb{C}} X}$.

$L \to S$ is a holomorphic line bundle.

$C = \sum n_i C_i \subset S$ a *divisor* in S: A linear combination of codimension 1 subvarieties (in this case curves).

$\mathcal{O}(C)$ is the line bundle with a section of zero order n_i along C_i.

$\chi(C) = \chi(S, \mathcal{O}(C))$ is the Euler characteristic.

$C \cdot D$ is the intersection form, computed as $\int_S [C] \cup [D]$.

$\mathrm{Pic}(S)$, resp. $\mathrm{Pic}^0(S)$, is the group of holomorphic line bundles under tensor product, resp. for which $c_1(L) = 0$.

$NS(S)$ is the *Neron-Severi group*: The subgroup of $H^2(S,\mathbb{Z})$ of classes $c_1(L)$ for some holomorphic line bundle L. Note there is an exact sequence $0 \to \mathrm{Pic}^0(S) \to \mathrm{Pic}(S) \to NS(S) \to 0$.

Note by Proposition 4.7, $c_1(\mathcal{O}(C)) = [C]$. Associated to a divisor is a critically important exact sequence which we will use all the time. Let X be a smooth projective variety, and let $D \subset X$ be a codimension 1 subvariety. Then there is an exact sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

where the map $\mathcal{O}(-D) \to \mathcal{O}$ is given by multiplication by the section $s_D \in H^0(X, \mathcal{O}(D))$ which vanishes along D. The image is the sheaf of functions which vanish along D and so the quotient is the sheaf of functions supported on D. It is a sheaf on X by declaring $\mathcal{O}_D(U) := \mathcal{O}(U \cap D)$, in a similar way to the skyscraper sheaves we've seen before.

More generally, we can tensor the above exact sequence of sheaves with a vector bundle, to get a sequence

$$0 \to \mathcal{E}(-D) \to \mathcal{E} \to \mathcal{E}\big|_D \to 0$$

which is still exact (since exactness is a local condition, and tensoring an exact sequence of $\mathcal{O}$-modules with a locally free module is still exact).

Proposition 7.1. *Let X be a smooth projective variety of dimension n. Every holomorphic line bundle $L = \mathcal{O}(C)$ for some divisor C.*

Proof. Let H be a *hyperplane section* of X: The intersection of X with a (generic) hyperplane in $\mathbb{CP}^N$. We require

Lemma 7.2 (Serre Vanishing Theorem)**.** *For any holomorphic vector bundle $\mathcal{E} \to X$, we have $h^i(X, \mathcal{E} \otimes \mathcal{O}(kH)) = 0$ for all $i > 0$, $k \gg 0$.*

Proof. The proof requires some analysis involving Kahler identities, see Huybrechts *Complex Geometry: An Introduction*, Proposition 5.2.7. □

We now apply the Hirzebruch-Riemann-Roch theorem. By multiplicativity of the Chern character,

$$\chi(L \otimes \mathcal{O}(kH)) = \int_X \mathrm{ch}(L)\mathrm{ch}(\mathcal{O}(H))^k \mathrm{td}(X).$$

Let $h = [H] \in H^2(X, \mathbb{Z})$. Then $\mathrm{ch}(\mathcal{O}(H))^k = 1 + kh + k^2h^2/2! + \cdots$. For $k \gg 0$, the dominant term is $k^{\dim X} h^{\dim X}/(\dim X)!$ because $h^{\dim X}$ is the intersection number of $\dim X$ hyperplanes in $\mathbb{CP}^N$ with X, which is positive. (By definition, it is the *degree* of X). So $\chi(L \otimes \mathcal{O}(kH)) \sim k^{\dim X} h^{\dim X}/(\dim X)!$ has order $\dim X$ polynomial growth in k. Thus, $h^0(L \otimes \mathcal{O}(kH)) > 0$ for large k. Let $D = \mathrm{div}(s)$ for $s \in H^0(L \otimes \mathcal{O}(kH))$ a non-zero section. Then $C = kH - D$ is a divisor for which $L = \mathcal{O}(C)$. □

Remark 7.3. A bit more work shows that we can ensure that sections of $L \otimes \mathcal{O}(kH)$ defines a projective embedding, so in fact, every line bundle (or divisor) is a difference of two two very ample divisors. By Bertini's theorem, these divisors may be chosen to be smooth.

Proposition 7.4. *Let C and D be curves sharing no component. Their intersection number can be computed in the following ways:*

$$C \cdot D = \deg \mathcal{O}(C)\big|_D = \sum_{p \in C \cap D} \mathrm{len}_p(C \cap D)$$

where $\mathrm{len}_p(C \cap D)$ *is the length at p of the scheme-theoretic intersection of C and D.*

Proof. We have $C \cdot D = \deg \mathcal{O}(C)\big|_D$ because

$$\int_S [C] \cup [D] = \int_C [D] = \int_C c_1(\mathcal{O}(D)).$$

We have an exact sequence of sheaves

$$0 \to \mathcal{O}_S(-C - D) \to \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_{C \cap D} \to 0$$

where the first two maps are multiplication by (s_D, s_C) and $(s_C, -s_D)^T$ and the last map is restriction.

The exactness at the second term follows from the fact that $\mathcal{O}_S(U)$ is a UFD for a contractible open set U and that in local trivializations, s_C, s_D share no common prime factor because C, D share no component. Exactness at the last term is the definition of the sheaf $\mathcal{O}_{C\cap D}$: It takes the value $\mathcal{O}_{C\cap D}(U) = \mathcal{O}(U)/\mathcal{O}(U)s_C + \mathcal{O}(U)s_D$ on a small open set U. Its support is the set $C \cap D$. By definition,

$$\textstyle\sum_{p\in C\cap D} \operatorname{len}_p(C\cap D) = h^0(\mathcal{O}_{C\cap D}).$$

Note $h^i(\mathcal{O}_{C\cap D}) = 0$ for all $i > 0$ because $C \cap D$ has dimension 0. By additivity of the Euler characteristic, it suffices to show that

$$\textstyle\sum_{p\in C\cap D} \operatorname{len}_p(C\cap D) = \chi(\mathcal{O}_S(-C-D)) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S).$$

Then the desired equality follows from the Riemann-Roch formula for algebraic surfaces, see Exercise 5.6. □

Remark 7.5. Alternatively, Proposition 7.4 can be used to prove Riemann-Roch for line bundles on algebraic surfaces—this requires showing that small holomorphic perturbations near p of C and D which intersect transversely turn the intersection point p into exactly $\operatorname{len}_p(C\cap D)$ intersection points.

Proposition 7.6 (Adjunction Formula)**.** *Let C be a smooth curve in S, then $\Omega^1_C = (\Omega^2_S \otimes \mathcal{O}(C))\big|_C$ and in particular,*

$$2g(C) - 2 = C \cdot (C + K).$$

Proof. Associated to any local section $s \in \Omega^2_S \otimes \mathcal{O}(C)(U)$ we can construct a meromorphic 2-form $s/s_C \in \operatorname{Mero}\Omega^2_S(U)$ with (at worst) a first order pole along C. Let $p \in C$ and let $\gamma_p(r)$ be a small oriented loop in $S \setminus C$ around $p \in C$ of radius r. Then the *residue map* $\operatorname{res}_C \colon \Omega^2_S \otimes \mathcal{O}(C) \to \Omega^1_C$ is

$$\operatorname{res}_C(s/s_C) := \lim_{r\to 0} \frac{1}{2\pi i} \oint_{\gamma_p(r)} s/s_C \in \Omega^1(U\cap C).$$

This map is $\mathcal{O}_S$-linear and the kernel consists of the 2-forms with no pole along C. We conclude that there is an exact sequence of sheaves

$$0 \to K_S \xrightarrow{\cdot s_C} K_S \otimes \mathcal{O}(C) \to K_C \to 0$$

and so res_C is identified with the restriction map of line bundles. □

Remark 7.7. The same argument works in general for a smooth divisor D in any smooth complex manifold X. In a local holomorphic coordinate system $(x_1, \dots, x_n)$ where $D = V(x_1)$, a holomorphic top-form on X is of the form $s = f(x_1,\dots,x_n)\, dx_1 \wedge \cdots \wedge dx_n$ for some holomorphic function f. We conclude the following formula for the residue:

$$\operatorname{res}_D \frac{s}{s_D} = \lim_{r\to 0} \frac{1}{2\pi i} \oint_{|x_1|=r} f \frac{dx_1}{x_1} \wedge \cdots \wedge dx_n = f(0, x_2, \dots, x_n) dx_2 \wedge \cdots \wedge dx_n$$

Exercise 7.8. *Compute the genus of a smooth complex curve of degree d in $\mathbb{CP}^2$. Compute the genus of a smooth curve of bidegree (d, e) in $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Exercise 7.9. *Use the local formula for the residue in Remark 7.7 to check that the residue map res_D is well-defined (independent of coordinate chart) and $\mathcal{O}$-linear.*

We haven't discussed many examples of smooth compact 4-manifolds or compact complex surfaces yet. So here are some:

Example 7.10. Obviously, the simplest example is S^4.

Example 7.11. An important example is $\mathbb{CP}^2$. Since $\mathbb{CP}^2$ can be built from a single 0-, 2-, and 4-cell, we have

$$H^k(\mathbb{CP}^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, 4 \\ 0 & \text{if } k = 1, 3. \end{cases}$$

We know that the cup product on $H^2(\mathbb{CP}^2, \mathbb{Z})$ defines a unimodular lattice, whose Gram matrix is necessarily $[\pm 1]$. In fact, it is $[1]$: A generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$ is the class of a hyperplane h and $h^2 = 1$ because two general lines $\mathbb{CP}^1 \subset \mathbb{CP}^2$ intersect transversely at a single point. The canonical bundle $K = -3h$, and more generally $K_{\mathbb{CP}^n} = -(n+1)h$. The Hodge numbers are $h^{p,q} = 0$ unless $(p, q) = (0, 0), (1, 1), (2, 2)$ in which case $h^{p,q} = 1$.

Example 7.12. There is a smooth 4-manifold $\overline{\mathbb{CP}^2}$ which is the same as $\mathbb{CP}^2$ except that we have negated the orientation. The orientation at an intersection point of two hyperplanes is *not* compatible with the orientation on $\overline{\mathbb{CP}^2}$ and so the Gram matrix is $[-1]$.

Exercise 7.13. *Show that S^4 and $\overline{\mathbb{CP}^2}$ do not admit complex structures. Hint: Review the formulas we know about complex surfaces.*

Example 7.14. A *complete intersection* is the smooth intersection of $n-2$ hypersurfaces in $\mathbb{CP}^n$. For instance, the Fermat K3 surface is

$$S := \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{CP}^3.$$

It follows from the adjunction formula that $K_S = K_{\mathbb{CP}^3} \otimes \mathcal{O}_{\mathbb{CP}^3}(S)\big|_S = \mathcal{O}_S$ is a trivial bundle. So S has a global holomorphic non-vanishing 2-form. We conclude that $h^{2,0}(X) = h^0(S, \Omega^2) = 1$. By Hodge symmetry, $h^{0,2} = 1$. Next, we cite the following theorem:

Theorem 7.15 (Lefschetz hyperplane theorem)**.** *Let X be a projective variety of dimension $\dim X \geq 3$ and let Y be the smooth intersection of X with a hypersurface. Then the map $\pi_1(Y) \to \pi_1(X)$ is an isomorphism, as are the pullback maps $H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ for all $k < \dim Y$.*

We conclude that π_1 of a complete intersection X of dimension $\dim X > 1$ is zero, because $\pi_1(\mathbb{CP}^n) = 0$. For all $k \neq \dim X$, $H^k(X, \mathbb{Z})$ of a complete intersection is 0 for odd k, and $\mathbb{Z}$ for even k. In particular, $\pi_1(S) = 0$ and hence $h^1(X) = h^{1,0}(X) = h^{0,1}(X) = 0$. By Noether's formula,

$$2 = h^{0,0} - h^{0,1} + h^{0,2} = \chi(\mathcal{O}(S)) = \tfrac{1}{12}(K^2 + \chi_{\text{top}}(S)) = \tfrac{1}{12}(4 + h^{1,1}(S))$$

from which we conclude that $h^{1,1}(S) = 20$. This determines the Hodge diamond $h^{p,q}$ completely.

Let $T \to \mathbb{CP}^n$ denote the *tautological bundle* whose fiber over a point is the line in $\mathbb{C}^{n+1}$ which projectivizes to this point. Then T generates the Picard group of $\mathbb{CP}^n$ and we define $\mathcal{O}(-k) := T^{\otimes k}$. It is easy to show $\mathcal{O}(1) = \mathcal{O}(H)$ for a hyperplane $H \subset \mathbb{CP}^n$. For the following exercise, you may assume

$$H^i(\mathbb{CP}^n, \mathcal{O}(k)) = \begin{cases} 0 & \text{if } i \neq 0, n \\ \mathbb{C}[x_0, \dots, x_n]_k & \text{if } i = 0 \\ \mathbb{C}[x_0, \dots, x_n]_{-n-1-k} & \text{if } i = n \end{cases}$$

where $\mathbb{C}[x_0, \dots, x_n]_k$ denotes the vector space of degree k homogenous polynomials in $n+1$ variables.

Exercise 7.16. *Compute the Hodge diamond of a smooth hypersurface X of degree 5 in $\mathbb{CP}^3$.*

Example 7.17. Given two smooth surfaces B, F their product $B \times F$ is a smooth 4-manifold. More generally, there are surface fiber bundles S over a surface $F \to S \to B$ and some of these admit complex structures. For instance, if B and F are Riemann surfaces, the product works. An example is $\mathbb{CP}^1 \times \mathbb{CP}^1$, whose cohomology is easily computed to be

$$H^k(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 2 \\ \mathbb{Z} & \text{if } k = 0, 4 \\ 0 & \text{if } k = 1, 3. \end{cases}$$

The fiber classes $e := [pt \times \mathbb{CP}^1]$ and $f := [\mathbb{CP}^1 \times pt]$ intersect in a unimodular Gram matrix

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence generate H^2. We will see that there are other holomorphic $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$, called *Hirzebruch surfaces* $\mathbb{F}_n$. They are constructed by projectivizing the fibers of the rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}(n) \to \mathbb{CP}^1$.

Example 7.18. Let $\Gamma \subset PU(1,2)$ be a discrete, torsion-free subgroup of the projectivized group of Hermitian isometries of $\mathbb{C}^{1,2}$. Then Γ acts freely on *complex hyperbolic space*

$$\mathbb{P}\{v \in \mathbb{C}^{1,2} \,|\, v \cdot v > 0\} = \mathbb{B}^2 \subset \mathbb{CP}^2$$

which is a complex ball of dimension 2. In some special cases, Γ acts cocompactly, giving a compact complex surface $\mathbb{B}^2/\Gamma$, called a *ball quotient*. Similarly, let $\Gamma \subset PO(1,4)$ be a discrete, torsion-free subgroup of the projectivized group of Lorentzian isometries of $\mathbb{R}^{1,4}$. Then Γ acts freely on *real hyperbolic space*

$$\mathbb{H}^4 := \mathbb{P}\{v \in \mathbb{R}^{1,4} \,|\, v \cdot v > 0\}.$$

If it acts cocompactly, the quotient $\mathbb{H}^4/\Gamma$ is a compact *hyperbolic manifold*.

8. Blow-ups and blow-downs

We now describe a very important operation, which can be defined on either a smooth 4-manifold or a complex surface, called *blowing up.* First we describe the complex manifold case.

Definition 8.1. Let S be a complex surface, $p \in S$ a point. Let (x, y) be local holomorphic coordinates in a neighborhood $U \ni p = (0, 0)$ and set $U^* = U - \{p\}$. Consider the holomorphic map

$$U^* \to U \times \mathbb{CP}^1$$
$$(x, y) \mapsto ((x, y),\ [x : y]).$$

We define Bl_pU to be closure of the image of U^*. There is a holomorphic map $\pi\colon Bl_pU \to U$ projecting to the first coordinate and we define the *exceptional curve* to be $\pi^{-1}(p)$. Note that $\pi : Bl_pU - E \to U^*$ is an isomorphism. The *blow-up* of Bl_pS is defined to be the union of Bl_pU and $S - \{p\}$ along the common open set U^*.

Just as in the local case, we have a map $\pi : Bl_pS \to S$ for which π defines an isomorphism from $Bl_pS - E \to S - \{p\}$ and $E \mapsto \{p\}$.

Proposition 8.2. *As an operation on smooth manifolds, $Bl_pS = S\#\overline{\mathbb{CP}^2}$ is the connect-sum with a copy of the complex projective plane, with its negated orientation, and E represented by a line in $\overline{\mathbb{CP}^2}$.*

Proof. The proposition follows if we can show $\overline{\mathbb{CP}^2} \setminus B^4$ is diffeomorphic to $\pi^{-1}(U)$ with U a contractible holomorphic coordinate chart as above.

The blow-up $Bl_{(0,0)}\mathbb{C}^2$ is visibly the total space of the tautological line bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$, with E the zero-section. A Hermitian metric on $\mathcal{O}(-1)$ gives an isomorphism $\mathcal{O}(-1) \to \overline{\mathcal{O}(1)}$ of complex line bundles. Thus $Bl_{(0,0)}\mathbb{C}^2$ has an orientation-reversing diffeomorphism to the total space of $\mathcal{O}(1)$, with E being identified to the zero section. Finally, $\mathcal{O}(1)$ can be identified with the lines in $\mathbb{CP}^2$ through x, minus x, so a neighborhood of the zero section is the complement of a ball in $\mathbb{CP}^2$ containing x. $\square$

Corollary 8.3. $E^2 = -1$.

Remark 8.4. An alternate topological construction of the blow-up is to excise a 4-ball B^4, then collapse the boundary $\partial B^4 = S^3$ along the Hopf fibration $S^3 \to S^2$, through it is less clear from this perspective why the result is a manifold. The resulting collapsed sphere is identified with E.

Definition 8.5. Let $C \subset S$ be a complex curve, or more generally an effective divisor. The *strict transform* in $\widehat{C} \subset Bl_p(S)$ is the closure of $\pi^{-1}(C - \{p\})$ in $Bl_p(S)$. The *pullback* π^*C is defined as follows: If f is a local holomorphic function for which $C = V(f)$, then π^*C is the divisor described by the vanishing locus of the function $\pi^*f = f \circ \pi$.

Remark 8.6. When C does not contain p, its strict transform and its pullback are equal.

Example 8.7. Let $L \subset \mathbb{C}^2$ be a line through the origin in $\mathbb{C}^2$, i.e. $L = \{(\lambda x_0, \lambda y_0) : \lambda \in \mathbb{C}\}$ Then, the strict transform $\widehat{L} \subset Bl_{(0,0)}\mathbb{C}^2$ is isomorphic to L, and passes transversely through a unique point $((0,0),[x_0 : y_0]) \in E$ and so we can think of E as "separating out the tangent lines" through the point $(0,0)$. On the other hand the pullback π^*L is the vanishing locus of the function $y_0x - x_0y$ on $Bl_{(0,0)}\mathbb{C}^2 \subset \mathbb{C}^2 \times \mathbb{CP}^1$. By an easy local coordinate computation, this function vanishes to first order along E. So $\pi^*L = L + E$.

Exercise 8.8. *Show that $[\pi^*C] = \pi^*[C]$ i.e. the pullback of a divisor represents the pullback of the fundamental class of that divisor.*

Warning: The pullback is well-defined on cohomology, but the strict transform isn't! For instance, take two lines L_1 and L_2 in $\mathbb{CP}^2$, the first one passing through p and the second one not. Then

$$[\widehat{L}_2] = \pi^*[L_2] = \pi^*[L_1] = [\widehat{L}_1] + [E]$$

Definition 8.9. The *multiplicity* $m_p(C)$ of an effective divisor C at a point p is the intersection number of C with a generic line passing through p, in a local coordinate chart.

Proposition 8.10. *We have $H^2(Bl_p(S),\mathbb{Z}) = \pi^*H^2(S,\mathbb{Z}) \oplus \mathbb{Z}E$ and a decomposition $\pi^*C = \widehat{C} + m_p(C)E$.*

Proof. As an abelian group, the decomposition of $H^2(Bl_p(S),\mathbb{Z})$ follows almost immediately from the Mayer-Vietoris exact sequence. Next we observe that π^* is an isometry for the intersection form: Given real surfaces Σ_1 and Σ_2 in S, we may perturb them to ensure they intersect transversely and neither passes through p. Then the signed sum of intersection points between Σ_1 and Σ_2 is unchanged by pullback, since in this case, $\pi^*[\Sigma_i] = [\pi^{-1}(\Sigma_i)]$. By perturbing, we also see that $\mathbb{Z}E$ is orthogonal to $\pi^*H^2(S,\mathbb{Z})$.

Let $f(x,y) = \sum a_{ij}x^iy^j$ be a series expansion about p of a holomorphic function f satisfying $V(f) = C$. Then $m_p(C)$ is the degree of the lowest order monomial appearing in f. By making a linear change of coordinates, we may as well assume this lowest order term is $x^{m_p(C)}$. Taking local coordinates (x,μ) with $\mu = y/x$ on the blow-up Bl_pS, we see that the largest power of $\pi^*f(x,y) = f(x,x\mu)$ divisible by x is $m_p(C)$ and hence $V(\pi^*f)$ contains E with multiplicity exactly $m_p(C)$. □

Corollary 8.11. *We have $\widehat{C}^2 = C^2 - m_p(C)^2$.*

Remark 8.12. Let Σ be a real surface smoothly embedded in a complex surface S and consider the blow-up $Bl_p(S)$. By isotoping Σ, we can ensure that the tangent space $T_p\Sigma$ is a complex-linear subspace of T_pS, so we can still define the notion of a strict transform $\widehat{\Sigma}$.

Exercise 8.13. *Consider the cubic curves $C^{\text{node}} := V(y^2 - x^3 - x^2) \subset \mathbb{C}^2$ and $C^{\text{cusp}} := V(y^2 - x^3) \subset \mathbb{C}^2$. Describe the strict transforms of these curves* *after blowing up at the origin. Consider the projective closures in $\mathbb{CP}^2$: This means homogenize these cubics using powers of z, and consider the vanishing locus of the homogenized cubic in projective coordinates $[x : y : z]$. What are the self-intersection numbers of the strict transforms?*

Note that in the smooth world, Bl_pS and Bl_qS are the same thing: They are both connect sums of S with $\overline{\mathbb{CP}^2}$. On the other hand, in the word of complex manifolds, they may be different. For instance, consider the blow-up $Bl_{p,q,r}\mathbb{CP}^2$ at three distinct points of $\mathbb{CP}^2$. If these points are collinear, then the strict transform $\widehat{L} = \pi^*L - E_1 - E_2 - E_3$ of the complex line L passing through them has self-intersection $\widehat{L}^2 = -2$. But deforming the points slightly, the points are no longer collinear, and this cohomology class will not be represented by any *holomorphic* curve.

Proposition 8.14. *Consider the blow-up $\pi\colon \widehat{S} = Bl_p(S) \to S$ at a point. We have $K_{\widehat{S}} = \pi^*K_S + E$.*

Proof. Recall that the sheaf K_S assigns to an open set $U \mapsto K_S(U)$ the holomorphic 2-forms on U. Near p, the holomorphic line bundle K_S has a local frame of the form $\omega_p = dx\, dy$. Thus, in local coordinates $(x, \mu = y/x)$ on the blow-up, we have

$$\pi^*(dx\, dy) = dx\, d(x\mu) = x\, dx\, d\mu$$

which vanishes to first order along E. Conversely, a section of $K_{\widehat{S}}$ vanishing to first order along E is the pullback of a section of K_S. We conclude that there is an isomorphism

$$\begin{aligned} \pi^*K_S &\cong \text{the subsheaf of sections of } K_{\widehat{S}} \text{ vanishing along } E \\ &\cong K_{\widehat{S}} \otimes \mathcal{O}(-E) \end{aligned}$$

where the isomorphism from the second line to the first is to multiply by the global section s_E of $\mathcal{O}(E)$ vanishing along E. □

Next, we discuss the reverse to blowing up, algebraically and smoothly.

Theorem 8.15 (Castelnuovo Contractibility Criterion)**.** *Let S be a smooth complex surface and let $E \subset S$ be a holomorphic $\mathbb{CP}^1$ for which $E^2 = -1$. Then there is a holomorphic map $\pi\colon S \to \overline{S}$ of smooth complex surfaces which is an isomorphism away from E and contracts E to a point p. We call π the* blow-down *of E.*

Exercise 8.16 (For the advanced algebraically-minded student)**.** *Prove the Castelnuovo contractibility criterion for projective surfaces, using the following strategy: Use the Serre vanishing theorem 7.2 to construct a line bundle $L = \mathcal{O}(H)$ associated to a hyperplane section, for which $H^i(S, L) = 0$ for $i > 0$. Define $a := L \cdot E > 0$. Consider the various short exact sequences*

$$0 \to \mathcal{O}(H + (k-1)E) \to \mathcal{O}(H + kE) \to \mathcal{O}(H + kE)\big|_E \to 0$$

for $k = 1, \dots, a$ and take their long exact sequences. Use these long exact sequences to show that the line bundle $\mathcal{O}(H+aE)$ has a section not vanishing along E, and that sections of $\mathcal{O}(H+aE)$ separate points and tangents away from E. Conclude that there exists a map $S \to \overline{S}$ to a possibly singular surface which is an isomorphism away from E and contracts $E \mapsto p$ to a point. Finally, show that $\overline{S}$ is smooth by identifying the Zariski cotangent space m_p/m_p^2 with the global sections of $\mathcal{O}(-E)/\mathcal{O}(-2E) = \mathcal{O}(-E)\big|_E$.

Similarly, given a smooth 2-sphere $E \subset X$ in an oriented smooth 4-manifold for which $E^2 = -1$, we have a blow-down $X \to \overline{X}$ to a smooth manifold which is differentiable, contracts E to a point, and is a diffeomorphism elsewhere. It can be constructed by noting that a tubular neighborhood of E is an oriented disc bundle over S^2. Such disc bundles are uniquely classified by an integer, equal to the self-intersection of the zero-section. So a tubular neighborhood of E is diffeomorphic to a neighborhood of the zero-section of the tautological bundle $\mathcal{O}(-1) \to \mathbb{P}^1$, which has an identification $\mathcal{O}(-1) = Bl_{(0,0)}\mathbb{C}^2$ and thus a blow-down to $\mathbb{C}^2$.

Remark 8.17. Unlike in the complex world, in the smooth world, we can also blow down a smooth 2-sphere $E \subset X$ for which $E^2 = -1$. For instance, reverse the orientation on X so that $E^2 = -1$, then blow down, then reverse orientation again. Smoothly, there is no reason why we should be able to undo the operation $X \mapsto X\#\overline{\mathbb{CP}^2}$ but not $X \mapsto X\#\mathbb{CP}^2$. So fair **warning** to the topologists: The second operation, and its inverse, have no complex-analytic meaning.

Exercise 8.18. *Let $E \subset S$ be a holomorphic $\mathbb{CP}^1$ in a complex surface S for which $E^2 = 1$. Prove that there is no operation in the category of complex manifolds which gives, as a 4-manifold, the smooth blow-down of E.*

9. Spin and Spin^c groups

We now return to some differential geometry. But there is quite a bit of linear algebra with which you may be unfamiliar. So we review here the theory of Clifford algebras. Let V, $||\cdot||$ be an inner product space. We will usually assume that the inner product is positive-definite.

Definition 9.1. The *Clifford algebra* $Cl(V)$ is the quotient of the tensor algebra $T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$ by the two-sided ideal generated by all elements of the form $v \otimes v + ||v||^2 1$.

Since the ideal is generated by even elements of $T(V)$, the Clifford algebra inherits a $\mathbb{Z}_2$-grading: $Cl(V) = Cl_0(V) \oplus Cl_1(V)$. Let $\epsilon\colon Cl(V) \to Cl(V)$ be the map which acts on $Cl_i(V)$ by multiplication by $(-1)^i$. It is an algebra homomorphism. Choosing an orthonormal basis $(e_1, \dots, e_n)$ of V, we get a basis of $Cl(V)$ give by $e_{i_1} \cdots e_{i_k}$ with $i_1 < \cdots < i_k$. Note that $e_i^2 = -1$ and $e_ie_j = -e_je_i$.

Proposition 9.2. *Consider the degree filtration on $Cl(V)$ induced by the degree filtration on $T(V)$. Its associated graded $\operatorname{gr} Cl(V) = \bigwedge^\bullet V$ is canonically isomorphic to the exterior algebra on V.*

Proof. The top degree part of $v \otimes v + ||v||^2 1 = 0$ is simply $v \otimes v = 0$ which generates the relations of the exterior algebra (i.e. $v \wedge v = 0$). So $\bigwedge^\bullet V$ is certainly a quotient of $\operatorname{gr} Cl(V)$, with $e_{i_1} \cdots e_{i_k}$ mapping to the (residue of) $e_{i_1} \wedge \cdots \wedge e_{i_k}$. But since $Cl(V)$ and $\operatorname{gr} Cl(V)$ have the same dimension, equal to the dimension of $\bigwedge^\bullet V$, we conclude that no quotient is taken. □

Example 9.3. $Cl(\mathbb{R}) = \mathbb{R}[x]/(x^2+1) = \mathbb{C}$. $Cl_0(\mathbb{R}) = \mathbb{R}$ and $Cl_1(\mathbb{R}) = i\mathbb{R}$.

$$Cl(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_1e_2$$

subject to the relations $e_1^2 = e_2^2 = -1$ and $e_3 := e_1e_2 = -e_2e_1$. Then $e_3^2 = e_1e_2e_1e_2 = -e_1^2e_2^2 = -1$. So we see $Cl(\mathbb{R}^2) = \mathbb{H}$ are the quaternions. $Cl_0(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}e_3$ is a copy of $\mathbb{C} \subset \mathbb{H}$.

Exercise 9.4. *Identify the algebra $Cl(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}$. Hint: You can determine the splitting by finding an idempotent element $x \in Cl(\mathbb{R}^3)$. Then identify the images of x and $1 - x$ with copies of $\mathbb{H}$.*

Proposition 9.5. *We have $Cl(V) = Cl_0(V \oplus \mathbb{R})$ for any V.*

Proof. Letting e be a unit vector in $\mathbb{R}$, the isomorphism is given by $v_0 + v_1 \mapsto v_0 + v_1 e$. It is easy to see this is an isomorphism of algebras. □

An important corollary for us is:

Corollary 9.6. $Cl_0(\mathbb{R}^4) = \mathbb{H} \oplus \mathbb{H}$.

Definition 9.7. The group $\mathrm{Pin}(V)$ is the subgroup of the group of units of $Cl(V)$ generated by elements $v \in V$ with $||v||^2 = 1$. The subgroup $\mathrm{Spin}(V)$ is the intersection of $\mathrm{Pin}(V)$ with $Cl_0(V)$.

Example 9.8. We claim that $\mathrm{Spin}(3) = SU(2)$ is the group of unit quaternions. To see why, first note that $\mathrm{Spin}(3) \subset Cl_0(\mathbb{R}^3) = Cl(\mathbb{R}^2) = \mathbb{H}$ is some subset of the quaternions. For instance,

$$Cl_0(V) = \mathbb{R} \oplus \mathbb{R}e_1e_2 \oplus \mathbb{R}e_2e_3 \oplus \mathbb{R}e_3e_1$$

and $\mathrm{Spin}(3)$ is generated by all products vw with $v, w \in \mathbb{R}^3$ satisfying $||v|| = ||w|| = 1$. Fixing $v = e_1$ and varying over all choices of w in the unit circle of $\mathbb{R}e_1 \oplus \mathbb{R}x$, $x \in \mathbb{R}^3$ gives all 1-parameter subgroups $S^1 \subset S^3$ of the unit quaternions. These subgroups cover all of S^3. Conversely, by choosing a basis with $v = e_1$ we see that anything of the form vw is a unit quaternion, which is already closed under multiplication.

Example 9.9. We claim that $\mathrm{Spin}(4) = SU(2) \times SU(2)$. Note

$$\mathrm{Spin}(4) \subset Cl_0(\mathbb{R}^4) = Cl(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}.$$

For any copy of $\mathbb{R}^3 \subset \mathbb{R}^4$, the resulting identification $Cl_0(\mathbb{R}^4) = Cl(\mathbb{R}^3)$, gives an inclusion $SU(2) = \text{Spin}(3) \hookrightarrow \text{Spin}(4)$ and a direct computation shows that these copies of $SU(2)$ generate all of $S^3 \times S^3 \subset \mathbb{H} \oplus \mathbb{H}$.

Here is an alternative, though related proof that $SU(2) \times SU(2)$ is the universal cover of $SO(4)$: Identify $SU(2)$ with the unit quaternions. Then we have an action of "left and right copies" $SU(2)_L \times SU(2)_R$ on $\mathbb{H} = \mathbb{R}^4$ by left- and right-multiplication of unit quaternions. These actions preserve the quaternionic norm and surject onto $SO(4)$. The subgroup acting trivially is the diagonal copy $\pm(1,1)$ of $\mathbb{Z}_2$.

Proposition 9.10. *There is a homomorphism* $\text{Spin}(n) \to SO(n)$ *with kernel* $\mathbb{Z}_2$*. Furthermore if* $n \geq 3$*, then* $\text{Spin}(n)$ *is the universal cover of* $SO(n)$*.*

Sketch. We claim that the action of $\text{Spin}(V)$ via conjugation on $Cl(V)$ preserves $V \subset Cl(V)$ and acts in an orientation-preserving manner. Let $||u|| = 1$, $u \in V$ so that $u^{-1} = -u$. Then conjugation acts by

$$x \mapsto uxu^{-1} = -uxu = -u(-ux - 2x \cdot u) = -x + 2u(x \cdot u) = -R_u x$$

where $R_u \in O(V)$ is the reflection in the hyperplane orthogonal to u. So there is a homomorphism $\text{Pin}(V) \to O(V)$ sending generators u, $||u|| = 1$ to negated reflections. This gives a homomorphism $\text{Spin}(V) \to SO(V)$ sending generators $u_1 u_2 \mapsto (-R_{u_1})(-R_{u_2}) = R_{u_1} R_{u_2}$ to the composition of two reflections. Every element of $SO(V)$ is a composition of reflections, and thus $\text{Spin}(V) \to SO(V)$ is surjective.

The kernel is the intersection of the center of $Cl(V)$ with $\text{Spin}(V)$, since u acts trivially on $V \subset Cl(V)$ iff it commutes with any element of V iff it commutes with all of $Cl(V)$. When n is odd, the center is generated by $1, e_1 \cdots e_n$ and when n is even, it is generated by 1 and so in either case, the intersection of the center with $Cl_0(V)$ is $\mathbb{R}$. Thus, the kernel is $\{\pm 1\}$.

It is not hard to show that $\text{Spin}(n) \to SO(n)$ is a non-trivial covering map. Since $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$, we conclude that $\text{Spin}(n)$ is in fact the universal cover. □

Remark 9.11. Here is one way to think about the group $\text{Pin}(n)$. A reflection is determined by either of two unit vectors $\pm u$. In the group $O(n)$ we don't distinguish the isometry of $\mathbb{R}^n$ gotten by reflecting in u vs $-u$. But in $\text{Pin}(n)$, we do make this distinction. There are two ways to make this distinction: A smart way and a stupid way. The stupid way (which gives $\mathbb{Z}_2 \times O(n)$) is to have the square of the reflection in u be the identity and the smart way (which gives $\text{Pin}(n)$) is to have the square of the reflection in u be a non-identity central element. Then $\text{Spin}(n)$ are the elements of $\text{Pin}(n)$ whose underlying isometry is orientation-preserving.

Definition 9.12. The group $\text{Spin}^c(V)$ is the subgroup of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by $\text{Spin}(V)$ and the unit circle in $\mathbb{C}$. Since the scalars in $\text{Spin}(V)$ are $\{\pm 1\}$, we have an isomorphism

$$\text{Spin}^c(V) = \text{Spin}(V) \times_{\{\pm 1\}} S^1.$$

Note that the quotient of $\mathrm{Spin}^c(V)$ by its center is also $SO(V)$. We now start to manifold versions of the above linear algebraic constructions.

Example 9.13. $\mathrm{Spin}^c(4) \subset U(2) \times U(2)$ is the subgroup of pairs (α, β) for which $\det(\alpha) = \det(\beta) \in S^1$.

Definition 9.14. Let (X, g) be an oriented smooth Riemannian manifold of dimension n so that TX is the associated bundle of a principal $SO(n)$-bundle $P \to X$. A *spin structure* (resp. spin^c structure) on X is a lift of P to a principal $\mathrm{Spin}(n)$-bundle $\widetilde{P}$.

Proposition 9.15. *X admits a spin structure if and only if $w_2(X) = 0$. When $w_2(X) = 0$, the spin structures is a torsor over $H^1(X, \mathbb{Z}_2)$.*

Proof. Consider the transition functions $t_{ij} : U_i \cap U_j \to SO(n)$ of the principal $SO(n)$-bundle P. Lift the transition functions arbitrarily

$$\widetilde{t}_{ij} : U_i \cap U_j \to \mathrm{Spin}(n).$$

There are two such lifts for every i, j. Define

$$(\widetilde{t}_{ijk}) = (\widetilde{t}_{ki} \circ \widetilde{t}_{jk} \circ \widetilde{t}_{ij}) \in C^2_{\mathfrak{U}}(X, \underline{\mathbb{Z}_2})$$

Then $\partial(\widetilde{t}_{ijk}) = 0$ is a Cech cocycle and choosing other lifts of t_{ij} change $(\widetilde{t}_{ijk})$ by a coboundary. So we get a well-defined cohomology class $[(\widetilde{t}_{ijk})] \in H^2(X, \mathbb{Z}_2)$ which equals $w_2(TX) = w_2(X)$. More generally, this procedure computes $w_2(\mathcal{E})$ for any vector bundle.

So $w_2(X) = 0$ if any only if $(\widetilde{t}_{ijk})$ is a coboundary. In this case, there exist lifts $\widetilde{t}_{ij}$ for which all $\widetilde{t}_{ijk} = 1$ i.e. the functions $\widetilde{t}_{ij}$ define a lifting to a $\mathrm{Spin}(n)$ bundle. Fix one such lift. The other lifts satisfying the cocycle condition are identified with the cocycles $(s_{ij}) \in C^2_{\mathfrak{U}}(X, \underline{\mathbb{Z}_2})$ via $(\widetilde{t}_{ij}) \mapsto (s_{ij}\widetilde{t}_{ij})$. Finally, two such lifts give isomorphic spin bundles if and only if they differ by the coboundary of a 1-chain, corresponding to post-composing the trivializations over U_i with an element of $\mathbb{Z}_2$. □

Theorem 9.16. *Assume n is even. Then $Cl(V)$ has a unique nontrivial, irreducible, finite dimensional complex representation S of dimension $2^{n/2}$, called the* spin representation. *In fact, $Cl(V) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}(S)$.*

Example 9.17. We have that $Cl(\mathbb{R}^2) = \mathbb{H}$ which has an irreducible complex representation of dimension 2 by sending

$$1 \mapsto I \qquad i \mapsto \sigma_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$j \mapsto \sigma_2 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad k \mapsto \sigma_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We call σ_i the *Pauli matrices*. Theorem 9.16 actually follows inductively from this base case. At the next step of the induction, we have an isomorphism $Cl(\mathbb{R}^4) = \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ by sending

$$\begin{aligned} e_1 &\mapsto \gamma_1 := 1 \otimes \sigma_1 \\ e_2 &\mapsto \gamma_2 := 1 \otimes \sigma_2 \\ e_3 &\mapsto \gamma_3 := \sigma_1 \otimes i\sigma_3 \\ e_4 &\mapsto \gamma_4 := \sigma_2 \otimes i\sigma_3. \end{aligned}$$

These 4×4 matrices γ_i are called the *Dirac matrices*.

Exercise 9.18. *Verify that the above gives a representation of $Cl(\mathbb{R}^4)$. Write down the next step of the induction to identify*

$$Cl(\mathbb{R}^6) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2).$$

Hint: Figure out how to continue the above pattern.

Proposition 9.19. *Suppose n is even. As a $Cl_0(V)$-representation, S splits into a direct sum $S = S^+ \oplus S^-$ of two irreducible representations, and $Cl_1(V)$ acts by switching these two summands.*

Proof. Consider the element

$$\omega_{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n \in Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$$

which satisfies $\omega_{\mathbb{C}}^2 = 1$. Since $\omega_{\mathbb{C}}$ squares to the identity, there is an eigenspace decomposition $S = S^+ \oplus S^-$ into ± 1 eigenspaces for the action of $\omega_{\mathbb{C}}$ on S. Note that $\omega_{\mathbb{C}}$ commutes with $Cl_0(V) \otimes_{\mathbb{R}} \mathbb{C}$ but anti-commutes with $Cl_1(V) \otimes_{\mathbb{R}} \mathbb{C}$ because n is even. In other words, $Cl_0(V) \otimes_{\mathbb{R}} \mathbb{C}$ acts by preserving S^+, S^- individually whereas $Cl_1(V) \otimes_{\mathbb{R}} \mathbb{C}$ acts by switching S^+, S^- because $\omega_{\mathbb{C}}(g_0 + g_1) = (g_0 - g_1)\omega_{\mathbb{C}}$. □

Remark 9.20. Given a $\mathbb{Z}_2$-graded algebra, the above structure is sometimes called a *super-representation*.

Warning: The situation in dimension $n = 4$ (and also $n = 2$) is confusing because a number of coincidences occur. The dimension of S happens to equal $2^{4/2} = 2^2 = 4$. We emphasize: *This 4 has nothing to do with the fact that $n = 4$.* In general the dimension of the spin representation is some power of 2 which is much, much larger than n.

Note that $\mathrm{Spin}(4) \subset Cl_0(\mathbb{R}^4)$ is isomorphic to $SU(2) \times SU(2)$. Then S^+, S^- are two complex representations of $Cl_0(\mathbb{R}^4)$, both of dimension 2. Restricting, we get a representation of $SU(2) \times SU(2)$ on $S^+ \oplus S^-$ which is the obvious one.

10. Spin bundles and Dirac operators

We now wish to transport these structures to a manifold, taking $V = T_pX$. The fundamental issue

Definition 10.1. Let (X, g) be a Riemann manifold. Its *Clifford bundle* $Cl(X) := Cl(T^*X, g)$ is a vector bundle constructed by taking the Clifford algebra of each cotangent space with respect to the metric g. As a vector bundle, we have an isomorphism $Cl(X) = \bigwedge^\bullet T^*X$ but the ring structure is different.

Remark 10.2. $Cl(X)$ is the space of global sections of a sheaf Cl of associative rings; multiplication of two sections of $Cl(U)$ over an open set U is given by fiberwise Clifford multiplication.

Definition 10.3. Let $\widetilde{P} \to X$ be a lift of the principal $SO(n)$-bundle $P \to X$ for (T^*X, g) to a principal $\mathrm{Spin}(n)$-bundle. The associated bundle

$$\widetilde{P} \otimes_{\mathrm{Spin}(n)} S$$

is called the *spinor bundle* $\mathbb{S}$. The sections $H^0(X, \mathbb{S})$ are called *spinors*.

Proposition 10.4. *The spin bundle $\mathbb{S}$ has the natural structure of a module over the Clifford bundle $Cl(X)$.*

Proof. Consider the map $Cl(\mathbb{R}^n) \otimes S \to S$ given by the action of the Clifford algebra: $x \otimes s \mapsto x \cdot s$. This map is actually a homomorphism of $\mathrm{Spin}(n)$-representations by viewing $Cl(\mathbb{R}^n)$ as a representation via conjugation. So we get a map of associated bundles

$$\widetilde{P} \times_{\mathrm{Spin}(n)} (Cl(\mathbb{R}^n) \otimes S) \to \widetilde{P} \times_{\mathrm{Spin}(n)} S.$$

Tensoring commutes with taking associated bundles, giving an action of $\widetilde{P} \times_{\mathrm{Spin}(n)} Cl(\mathbb{R}^n)$ on $\mathbb{S}$. Finally, note that $\widetilde{P} \times_{\mathrm{Spin}(n)} Cl(\mathbb{R}^n) = Cl(\widetilde{P} \times_{\mathrm{Spin}(n)} \mathbb{R}^n) = Cl(T^*X)$ giving an action of the Clifford bundle on $\mathbb{S}$. □

As on any Riemannian manifold, P has a natural connection—the Levi-Civita connection. This connection lifts to $\widetilde{P}$: Since $\widetilde{P} \to P$ is a covering map, parallel transports lift uniquely. In turn, the associated bundle $\mathbb{S}$ admits a connection which we denote by ∇^{LC}.

Definition 10.5. A connection on the spin bundle $\mathbb{S}$ is *Clifford* if

$$\nabla(x \cdot s) = x \cdot \nabla(s) + \nabla(x) \cdot s$$

i.e. Clifford multiplication is parallel. $\nabla = \nabla^{LC}$ is an example.

Proposition 10.6. *The Clifford connections are a torsor over $\Omega^1(X)$.*

Proof. The difference $\alpha = \nabla - \nabla^{LC}$ of two connections defines a global section $\alpha \in \Omega^1 \otimes \mathrm{End}(\mathbb{S})(X)$. Since ∇^{LC} is Clifford, the condition that ∇ be Clifford implies $\alpha(e_i \cdot s) = e_i \cdot \alpha(s)$. We have $\mathrm{End}(\mathbb{S}) = Cl(X) \otimes_{\mathbb{R}} \mathbb{C}$ i.e. α is a Clifford bundle-valued one-form which commutes with the action of

Clifford multiplication. This is the same as a one-form valued in the center of the (complexified) Clifford bundle, which is just a one-form on X. $\square$

Definition 10.7. Let ∇ be a Clifford connection on $\mathbb{S}$ and let $s \in \mathbb{S}(U)$ so that $\nabla(s) \in \mathbb{S} \otimes \Omega^1(U)$. The *Dirac operator* $\partial\!\!\!/_\nabla : H^0(\mathbb{S}) \to H^0(\mathbb{S})$ is

$$\partial\!\!\!/_\nabla(s) := \sum_{e_i \text{ frame}} e_i \cdot \nabla_{e_i^*}(s)$$

where $e_i \cdot$ denotes Clifford multiplication by a frame $\{e_i\}$ of T^*U and $\{e_i^*\}$ is the dual frame of the tangent bundle TU. Write $\partial\!\!\!/ = \partial\!\!\!/_{\nabla^{LC}}$.

Exercise 10.8. *Show the Dirac operator is well-defined.*

Any Clifford connection $\nabla = \nabla^+ \oplus \nabla^-$ is a sum of connections w.r.t. the half-spin decomposition $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ because ∇^{LC} respects this decomposition, and wedging with $\alpha \in \Omega^1(X)$ sends each of $\mathbb{S}^\pm \to \mathbb{S}^\pm \otimes \Omega^1$. Since e_i lies in $Cl_1(X)$, we have that $\partial\!\!\!/_\nabla$ acts by switching sections $\mathbb{S}^+(U)$ and $\mathbb{S}^-(U)$.

Example 10.9. Consider $X = \mathbb{R}^4$ with the standard metric. Then $\mathbb{S}$ is a trivial complex vector bundle of rank 4 (with say frame s_1, s_2, s_3, s_4) and the connection $\nabla^{LC}_{e_i}$ is just the directional derivative. Following Example 9.17, the splitting $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{S}^+ \oplus \mathbb{S}^-$ is given by the ± 1-eigenspaces of

$$\gamma_5 := \omega_{\mathbb{C}} = -\gamma_1\gamma_2\gamma_3\gamma_4 = -\sigma_3 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

which are

$$\mathbb{S}^+ = \mathbb{C}(s_1 - s_4) \oplus \mathbb{C}(s_2 + s_3)$$
$$\mathbb{S}^- = \mathbb{C}(s_1 + s_4) \oplus \mathbb{C}(s_2 - s_3).$$

Write a section $\psi = \psi_1 s_2 + \psi_2 s_1 + \psi_3 s_3 + \psi_4 s_4$. Then the dirac equation $\partial\!\!\!/\psi = 0$ is the system of four first order differential equations

$$\gamma_1\psi_{x_1} + \gamma_2\psi_{x_2} + \gamma_3\psi_{x_3} + \gamma_4\psi_{x_4} = \vec{0}.$$

Taking the square of the Dirac operator, we get that

$$\partial\!\!\!/^2\psi = \sum_{i,j} \gamma_i\gamma_j\psi_{x_ix_j} = \sum_{i,j} -(e_i \cdot e_j)\psi_{x_ix_j} = -\sum_i \psi_{x_ix_i} = -\Delta\psi$$

is the coordinate-wise Laplacian. Here the key was that ∇_{e_i} and ∇_{e_j} are the partial derivatives, and hence commute.

Proposition 10.10. *The standard dirac operator $\partial\!\!\!/$ is self-adjoint.*

Note that this only makes sense by putting a hermitian metric on the spin bundle $\mathbb{S}$. There is a natural $\mathrm{Pin}(n)$-invariant metric unique up to scaling on S, and hence a metric also on $\mathbb{S}$. In particular, Clifford multiplication by a unit vector defines an isometry of the spin bundle.

Historical Tangent: We now discuss the historical relevance of the Dirac operator, which solved a number of outstanding problems in quantum mechanics. We note though that for actual applications to our spacetime, we should work with $Cl(\mathbb{R}^{3,1})$ rather than $Cl(\mathbb{R}^4)$. This sign changes alter structure constants defining the Clifford algebra, and affect the γ-matrices by various factors of i.

Recall from quantum mechanics the *Schrödinger equation*:

$$i\partial_t \psi = (-\tfrac{1}{2m}\Delta + V)\psi$$

where m is the mass, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, and V is the "potential"—an ambient real-valued function on $\mathbb{R}^{3,1}$ which heuristically describes "how unpleasant" it is for a particle to be at that point in spacetime. The physical interpretation of the *wavefunction* ψ is that at a fixed time t, the probability density that the particle is at a point x is given by $|\psi(x,t)|^2$.

Example 10.11. A simple example is the infinite potential well:

$$V(x,y,z,t) = \begin{cases} 0 & \text{if } |x|, |y|, |z| < 1 \\ \infty & \text{if otherwise.} \end{cases}$$

It is "infinitely unpleasant" for the particle to lie outside the cube centered at the origin, and so $\psi(x,y,z,t) = 0$ except on the inside of the cube.

When, as above, V doesn't depend on time, we can consider *stationary solutions* whose time-dependence is minimal: $\psi(x,y,z,t) = e^{-iEt}\psi(x,y,z)$ where E is call the *energy*. This gives, the so-called time-independent Schrodinger equation

$$E\psi = (-\tfrac{1}{2m}\Delta + V)\psi$$

with $\psi = \psi(x,y,z)$ which no longer depends on time.

Example 10.12. Consider a hydrogen atom, with one proton and one electron. The (time-independent) potential is $V = -1/r$, $r = \sqrt{x^2+y^2+z^2}$. Physically, this means the electron is *free* ($V \approx 0$) when $r \gg 0$: It acts as though it were in a vacuum with no potential at all. But when r is small, $V \ll 0$ which means that the electron likes being close to the proton—this is true because they have opposite charges.

Exercise 10.13. *Solve the time-independent Schrodinger equation for the hydrogen atom. Hint: Rewrite the Laplacian Δ in terms of spherical coordinates, and assume that $\psi(x,y,z) = R(r)\Phi(\phi)\Theta(\theta)$ factors into functions of the three spherical coordinates. You will find that $R(r)$ depends on an integer n called the* principal quantum number*, that $\Phi(\phi)$ depends on an $\ell \in \{0,\dots,n-1\}$ called the* orbital quantum number*, and that $\Theta(\theta)$ depends on an $m \in \{-\ell,\dots,\ell\}$ called the* magnetic quantum number*.*

Example 10.14. When $(n,\ell,m) = (1,0,0)$, the electron is in its lowest energy eigenstate. The probability density for the location of the electron is spherically symmetric with exponential radial decay.

The *Pauli exclusion principle* states that when there are multiple electrons, they cannot occupy the same eigenstates. Thus, assuming no interaction between electrons, there is a tendency for the electrons in an atom to successively fill eigenstates, in increasing order of energy. Up to a critical factor of 2, this approach is largely successful at predicting the periodic table, though the above exercise will show that the energy depends only on the principal quantum number n, which does not agree with experiment.

Problem 1: The Schrodinger equation is not *relativistically invariant*: The equation won't give the same wavefunction ψ under a change of coordinates in $SO(3,1)$, also called a Lorentz transformation. This is clear, for instance, because the Schrodinger equation is first order in t, but second order in the spatial variables x, y, z.

Problem 2: What explains the fact that each orbital with fixed (n, ℓ, m) can accommodate two, as opposed to one, electron?

These problems are simultaneously resolved by Dirac's equation, which also explains other physical phenomena such as antimatter. We now take the wavefunction $\psi \in H^0(\mathbb{R}^{3,1}, \mathbb{S})$ to be a spinor. An *electromagnetic vector potential* is a one-form $A \in H^0(\mathbb{R}^{3,1}, \Omega^1)$ usually considered up to exact one-forms. Writing $A = A_1 dx + A_2 dy + A_3 dz + \phi\, dt$ we have

$$F := dA = \frac{1}{2}\begin{pmatrix} dx & dy & dz & dt \end{pmatrix}\begin{pmatrix} 0 & -B_z & B_y & -E_x \\ B_z & 0 & -B_x & -E_y \\ -B_y & B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}$$

so that $\mathrm{curl}(\vec{A}) = \vec{B}$ and $\mathrm{div}(\phi) = \vec{E}$ are potentials for the magnetic and electric fields, respectively. Two of the four Maxwell equations are encoded by $dF = 0$, and the remaining two Maxwell equations are $d(*F) = J$ where $J = (j_x, j_y, j_z, q)$ is the four-current density, consisting of the classical current density $\vec{j}$ and the charge density q. In a vacuum $J = 0$, this gives the Maxwell equations as $dF = d(*F) = 0$ i.e. F is a harmonic 2-form.

Definition 10.15. The *Dirac equation* (for a particle of mass m in the presence of a potential one-form A) is the spinor equation

$$(i\partial\!\!\!/_A + m\omega)\psi = 0.$$

Here $\omega = \omega_{\mathbb{C}}$ acts on positive spinors by 1 and negative spinors by -1 and $\partial\!\!\!/_A := \partial\!\!\!/_\nabla$ where $\nabla = \nabla^{LC} + A$.

Remark 10.16. The Dirac equation is relativistically invariant essentially by the computation that $\partial\!\!\!/_A$ is well-defined, depending only on the metric, for any one-form $A \in \Omega^1(X)$ on a spin manifold X.

How does the Dirac equation help solve our periodic table problem? Essentially, in the non-relativistic limit, a solution to the dirac equation will satisfy $\psi^- = g \cdot \psi^+$ for some explicit element $g \in Cl(\mathbb{R}^{3,1})$ and so the equation reduces to an equation on the positive spinor ψ^+, called the *Pauli equation*. The eigenstates of the Pauli equation are given by the tensor product of an eigenstate of the usual Schrodinger equation with an eigenvector for a two-dimensional quantum system

$$\mathbb{C}v_{\uparrow} \oplus \mathbb{C}v_{\downarrow}.$$

with two eigenstates, usually called *spin up* and *spin down*, on which the Pauli matrices act. Thus, there is a fourth *spin quantum number*, equal to either $\pm\frac{1}{2}$ which allows for two electrons in each (n, ℓ, m) orbital type.

11. Rokhlin's theorem

Returning to the world of math, we now have the tools to prove:

Theorem 11.1 (Rokhlin's Theorem)**.** *Let X be a closed, oriented, spin 4-manifold. The signature is divisible by 16.*

Proof. The proof goes in two steps. Let $\partial\!\!\!/^{+} \colon \mathbb{S}^{+}(X) \to \mathbb{S}^{-}(X)$ be the restriction of the dirac operator to positive spinors. The first step is to show that $\mathrm{ind}(\partial\!\!\!/^{+}) = -\mathrm{sig}(X)/8$. The second step is to show that this index is always even. The symbol of the Dirac operator is

$$\begin{aligned} \sigma(\partial\!\!\!/) : \pi^*\mathbb{S} &\to \pi^*\mathbb{S} \\ s &\mapsto \textstyle\sum_i y_i e_i \cdot s \end{aligned}$$

or more canonically, at a point $(p, \alpha) \in T^*X$, the symbol acts on $\pi^*\mathbb{S}$ as the Clifford multiplication $\alpha \cdot -$. Away from the zero section $\alpha = 0$, this map is invertible, because its square $\alpha \cdot \alpha = -||\alpha||^2 1$ is. Suppose for now just that $\dim X = 2n$ is even.

Lemma 11.2. *The difference of the Chern characters of the spin bundles is $\mathrm{ch}(\mathbb{S}^+) - \mathrm{ch}(\mathbb{S}^-) = \prod_{i=1}^n (e^{x_i/2} - e^{-x_i/2})$ where $\pm x_i$ are the Chern roots of the complexified cotangent bundle.*

Proof. First, by (the real version of) the splitting principle we may assume

$$T^*X \otimes_{\mathbb{R}} \mathbb{C} = (L_1 \oplus \overline{L}_1) \oplus \cdots \oplus (L_n \oplus \overline{L}_n)$$

with the natural action of complex conjugation acting by switching sections of L_i and $\overline{L}_i$. We deform the metric on T^*X so that it is a direct sum of metrics on each summand $(L_i \oplus \overline{L}_i)_{\mathbb{R}}$ and the transition functions lie in

$$SO(2) \times \cdots \times SO(2) \subset SO(2n).$$

The spin structure corresponds to taking consistent square-roots for the transition functions for all L_i or equivalently finding complex line bundles M_i for which $M_i \otimes M_i = L_i$. With respect to this subgroup, the spin representation is the tensor product

$$\mathbb{S} = (M_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} M_n) \otimes_{\mathbb{R}} \mathbb{C} = (M_1 \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (M_n \otimes_{\mathbb{R}} \mathbb{C})$$

with $\omega_{\mathbb{C}}$ acting on each factor $M_1 \otimes_{\mathbb{R}} \mathbb{C} = M_1 \oplus \overline{M}_1 = M_1 \oplus M_1^{-1}$ by $(1, -1)$. These facts follow from the inductive description of the spin representation in even dimension $2n$. We conclude that

$$\mathbb{S}^+ = \bigoplus_{\substack{a_i \in \{\pm 1\} \\ \prod a_i = 1}} M_1^{a_1} \otimes \cdots \otimes M_n^{a_n} \text{ and } \mathbb{S}^- = \bigoplus_{\substack{a_i \in \{\pm 1\} \\ \prod a_i = -1}} M_1^{a_1} \otimes \cdots \otimes M_n^{a_n}.$$

Since $c_1(M_i) = x_i/2$ we conclude that $\mathrm{ch}(\mathbb{S}^{\pm})$ is the sum of all terms in the expansion of $\prod_{i=1}^n (e^{x_i/2} + e^{-x_i/2})$ with an even, resp. odd, number of minuses. We conclude that

$$\mathrm{ch}(\mathbb{S}^+) = \textstyle\prod_{i=1}^n (e^{x_i/2} + e^{-x_i/2}) \text{ and } \mathrm{ch}(\mathbb{S}^-) = \textstyle\prod_{i=1}^n (e^{x_i/2} - e^{-x_i/2}).$$

□

Now, we can apply Atiyah-Singer. We have by Lemma 11.2 that

$$\begin{aligned}
\mathrm{ind}(\not{\partial}^+) &= (-1)^n \int_X \frac{\mathrm{ch}}{\mathrm{eul}}(\mathbb{S}^+ - \mathbb{S}^-)\mathrm{td}(TX \otimes \mathbb{C}) \\
&= \int_X (\mathrm{ch}(\mathbb{S}^+) - \mathrm{ch}(\mathbb{S}^-)) \prod_{i=1}^n \frac{x_i}{(1 - e^{x_i})(1 - e^{-x_i})} \\
&= \int_X \prod_{i=1}^n \frac{x_i(e^{x_i/2} - e^{-x_i/2})}{(1 - e^{x_i})(1 - e^{-x_i})} = (-1)^n \int_X \prod_{i=1}^n \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}.
\end{aligned}$$

which vanishes unless n is even, i.e. $4 \mid \dim X$. When $n = 4$, we have

$$\mathrm{ind}(\not{\partial}^+) = \int_X (1 - x_1^2/24)(1 - x_2^2/24) = \int_X (-c_1^2 + 2c_2)/24 = -\mathrm{sig}(X)/8$$

where the last equality follows from Lemma 11.3.

Lemma 11.3 (Hirzebruch Signature Theorem)**.** *Let X be an oriented smooth 4-manifold. We have $c_1^2 - 2c_2 = p_1(X) = 3 \cdot \mathrm{sig}(X)$.*

Proof. We yet again apply the Atiyah-Singer theorem (so useful!). Consider the operator d and its adjoint d^* acting on $\Omega^{\bullet}(X)$. Note that $d + d^*$ is a self-adjoint operator. We define $\tau(\omega) := i^{p(p-1)+n} * \omega$ where $\dim X = 2n$.

Exercise 11.4. *Show $\tau^2 = 1$ and $d + d^*$ anticommutes with τ.*

We conclude that $d + d^*$ acts by permuting the ± 1-eigenspaces $\Omega^{\pm}(X)$ of τ. So we have a block form

$$d + d^* = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$$

for $D : \Omega^+(X) \to \Omega^-(X)$ the restriction of $d + d^*$. We claim that $\mathrm{ind}(D) = \mathrm{sig}(X)$. To see why, first note that

$$\begin{aligned}
\dim\ker(D) - \dim\mathrm{coker}(D) &= \dim\ker(D) - \dim\ker(D^*) \\
&= \dim\mathcal{H}^+(X) - \dim\mathcal{H}^-(X)
\end{aligned}$$

where $\mathcal{H}^{\pm}(X)$ are the ± 1-eigenspaces of harmonic forms under the action of τ. Consider the action of τ on $\mathcal{H}^{n+k} \oplus \mathcal{H}^{n-k}$, $k \neq 0$. Given a basis $\{\omega_i\}$ of $\mathcal{H}^{n-k}$ this space has a basis $\{\omega_i, \tau\omega_i\}$ and thus the dimensions of the $+1$ and -1 eigenspaces on the sum are equal.

When n is even $\tau\big|_{\mathcal{H}^n} = *$ and so the ± 1-eigenspaces are the self-dual and anti-self-dual harmonic middle forms. We have

$$0 < \int_X \alpha \wedge *\alpha = \text{eigenvalue}(*) \int_X \alpha \wedge \alpha$$

and so the ± 1-eigenspaces of $*$ represent positive- and negative-definite subspaces for the intersection form on $H^n(X, \mathbb{R})$.

It remains to compute the index of D using the Atiyah-Singer index theorem. We leave this as an rather difficult exercise. □

Exercise 11.5. *As in Lemma 11.2, use the splitting principle to compute* $\mathrm{ch}(\Omega^+) - \mathrm{ch}(\Omega^-)$. *Plug this into the index formula for D and expand the integrand to compute* $\mathrm{ind}(D)$ *in dimensions* $2n = 4, 8$.

We have finished the first step of the proof of Rokhlin's theorem. Next we must show $\mathrm{ind}(\partial\!\!\!/^+)$ is even. This follows if we can show that

$$\ker(\partial\!\!\!/^+) \text{ and } \mathrm{coker}(\partial\!\!\!/^+) = \ker(\partial\!\!\!/^-)$$

admit quaternionic structures. Then they have even complex dimension.

First, note the spin representations $S^{\pm}$ admit quaternionic structures in dimension 4. This is easy because the spin group $\mathrm{Spin}(4) = SU(2) \times SU(2) = S^3 \times S^3 \subset \mathbb{H} \oplus \mathbb{H} = S^+ \oplus S^-$ is the product of the unit quaternions with itself. More canonically, we have:

Exercise 11.6. *Show that there is a real structure on $\mathbb{C}^2$ (a complex-antilinear map $c : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying $c^2 = 1$) such that c commutes with the two Pauli matrices σ_1, σ_2. Show that there is a quaternionic structure on $\mathbb{C}^2$ (a complex-antilinear map $j : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying $j^2 = -1$) such that j anticommutes with the two Pauli matrices σ_1, σ_2. Conclude that $c \otimes j$ is a quaternionic structure on $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{S}$ which anticommutes with Clifford multiplication by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and commutes with $\omega_{\mathbb{C}} = -\sigma_3 \otimes \sigma_3$.*

From the exercise, we conclude that there is a quaternionic structure $J := c \otimes j$ on $\mathbb{S}$ which is compatible with the decomposition $\mathbb{S}^{\pm}$ into $\omega_{\mathbb{C}}$-eigenspaces and anticommutes with Clifford multiplication. Consider

$$L : s \mapsto \partial\!\!\!/(Js) + J\partial\!\!\!/(s).$$

Multiplying s by a real-valued function f, applying Liebniz, and using that $Jdf \cdot s + df \cdot Js = 0$ we get that L is $\mathcal{C}^{\infty}$-linear (over real functions). So L defines a complex anti-linear bundle map $\mathbb{S}^{\pm} \to \mathbb{S}^{\mp}$ which anticommutes with Clifford multiplication. But then L is, fiberwise, an intertwiner between S^+ and the complex conjugate of S^- as spin representations. This is absurd for $SU(2) \times SU(2)$ unless $L = 0$.

We conclude that $\partial\!\!\!/ J = -J\partial\!\!\!/$ and so J acts on $\ker(\partial\!\!\!/^{\pm})$. Thus, the two spaces in question have quaternionic structures, as desired. □

Remark 11.7. Rokhlin's theorem generalizes, see Ochanine, to a spin manifold with $\dim X \equiv 4 \bmod 8$.

Remark 11.8. If $H_1(X,\mathbb{Z})$ has no 2-torsion, then $w_2(X) = 0$ if and only if the intersection form on X is even. The condition on torsion is necessary, e.g. consider an Enriques surface. So in this setting, X is spin.

Theorem 11.9. *There is no smooth sphere representing the class $3h \in H^2(\mathbb{CP}^2,\mathbb{Z})$.*

Proof. Suppose there were such a sphere $\Sigma \subset \mathbb{CP}^2$. Consider the blow-up $S := Bl_{p_1,\dots,p_8}\mathbb{CP}^2$ at eight points lying on Σ and let $\widehat{\Sigma}$ be the strict transform. Then $\widehat{\Sigma}^2 = 1$ and so there is a smooth blow-down $S \to \overline{S}$ contracting $\widehat{\Sigma}$. The intersection form on $\overline{S}$ is that of $[\widehat{\Sigma}]^\perp \subset H^2(S,\mathbb{Z})$. Let $h, e_1, \dots, e_8$ be the basis of $H^2(S,\mathbb{Z})$ corresponding to the pullback of the hyperplane class, and the classes of the eight exceptional spheres. These form an orthonormal basis, with $h^2 = 1$ and $e_i^2 = -1$. We have

$$[\widehat{\Sigma}] = 3h - e_1 - \cdots - e_8$$

and so a direct computation shows that $[\widehat{\Sigma}]^\perp = H^2(\overline{S},\mathbb{Z})$ is an even unimodular lattice of signature profile $(0,8)$.

None of the blow-up or blow-down operations affect the fundamental group and so $\pi_1(\overline{S}) = \pi_1(\mathbb{CP}^2) = 0$. We conclude by Remark 11.8 that $\overline{S}$ admits a spin structure. But its signature -8 is not divisible by 16, which contradicts Rokhlin's theorem. Thus, no such sphere Σ exists. □

Exercise 11.10. *Using similar arguments, find as many degrees d as you can for which dh cannot be represented by a smooth sphere.*

12. Freedman's theorems

We now review (without proofs!) some important historical results about the classification of 4- and higher-dimensional manifolds.

Definition 12.1. An *oriented cobordism* between two oriented manifolds X, Y is an oriented manifold M whose boundary is $\partial M = X \sqcup \overline{Y}$. We say that M is an *h-cobordism* if $X, Y \hookrightarrow M$ are homotopy equivalences.

Theorem 12.2 (Smale)**.** *In the smooth category, any h-cobordism between simply connected manifolds of dimension $\dim X = \dim Y \geq 5$ is a product: $M = X \times I$.*

Theorem 12.3 (Freedman)**.** *In the topological category, any h-cobordism between 4-manifolds X, Y is a product: $M = X \times I$.*

Remark 12.4. We won't prove these theorems, but here is some indication as to how Smale's works in the smooth category. One begins with a Morse function $f : M \to [0,1]$ for which $f^{-1}(0) = X$ and $f^{-1}(1) = Y$. Let

$0 < c_0 < \cdots < c_n < 1$ be the critical values of f, each one the image of just one critical point. Choose constants $c_{i-1} < d_i < c_i$. The sequence

$$(X, X_1, \ldots, X_n, Y)$$

of manifolds defined by $X_i := f^{-1}[0, d_i]$ are related by successive *handle attachments*. The proof strategy is to cancel the handle attachments in pairs until none are left, and f has no critical points. Then the gradient flow of f will define a diffeomorphism. The h-corbordism assumption gives that the relative homotopy groups $\pi_k(M, X) = 0$ are trivial, roughly leading to the fact that handles pair up.

Where does the dimension bound come in? The key step in handle cancellation is to apply the Whitney trick: If two oriented submanifolds of complementary dimension in an ambient oriented manifold intersect transversely at two points of opposite sign, then they can be isotoped until they are disjoint. The problem is that the Whitney trick is only true in dimension at least 5. The miraculous result of Freedman is that is also works topologically in dimension 4.

Theorem 12.5 (Thom-Pontryagin)**.** *The pontryagin classes are oriented cobordism invariants. More precisely, the exceptional cohomology theory* Ω^{SO*} *for oriented cobordism satisfies* $\Omega^{SO*}(\mathrm{pt}) \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, \cdots]$*. This ring is generated by the cobordism classes* $[\mathbb{CP}^{2n}]$*,* $n > 0$*.*

The proof of the Thom-Pontryagin theorem follows from homotopy theory and a careful study of the stable normal bundle. All embeddings of X (or cobordisms from X to Y) into $\mathbb{R}^N$ are homotopic for large enough N. This ensures that the classifying map of the stable normal bundle to an appropriate classifying space eventually stabilizes in stable homotopy theory, and that cobordant manifolds give the same stable homotopy class.

So we get a ring homomorphism from the cobordism ring to the stable homotopy groups of the classifying space (strictly speaking the relative homotopy groups of the tangent space of the classifying space, relative to the zero section). Conversely, if the element of the stable homotopy group associated to the manifold X equals zero, it is possible, using the homotopy and transversality results, to construct a manifold M whose boundary is X.

Remark 12.6. Fixing a target T which is not a point, we declare $\Omega_p^{SO}(T)$ to be the smooth maps $f : X^p \to T$ from a closed oriented boundaryless p-manifold X^p, modulo the equivalence relation $(X^p, f) \sim (Y^p, g)$ if there is an oriented cobordism M^{p+1} between X^p and Y^q, together with a map $h \colon M^{p+1} \to T$ such that $h\big|_{X^p} = f$ and $h\big|_{Y^p} = g$. These equivalence classes form an abelian group with inversion given by reversing orientation on X^p and addition given by disjoint union.

More generally, if X^p is a manifold with boundary, then we declare the *boundary* of (X^p, f) to be $(\partial X^p, f\big|_{\partial X^p})$. This allows us to define a chain complex (or dually a cochain complex) whose homology (or cohomology) gives $\Omega_*^{SO}(T)$ (or $\Omega^{SO*}(T)$). Then Ω^{SO*} defines an extraordinary cohomology theory, i.e. it satisfies all Eilenberg-Steenrod axioms, excluding the condition that $\Omega^{SOp}(\mathrm{pt}) = 0$ for all $p \neq 0$.

Corollary 12.7. *The signature of an oriented $4n$-manifold X is a cobordism invariant. In particular, if $X = \partial M$ is a boundary of a $4n+1$-manifold, then* $\mathrm{sig}(X) = 0$.

Proof. This follows from our proof the Hirzebruch signature theorem and Exercise 11.5, which shows that $\mathrm{ch}(\Omega^+) - \mathrm{ch}(\Omega^-)$ is expressible in terms of $p_{i/2}(X) = (-1)^{i/2} c_i(TX \otimes \mathbb{C})$ [as are $\mathrm{eul}(X)$ and $\mathrm{td}(TX \otimes \mathbb{C})$ which appear in the Atiyah-Singer index theorem]. □

Consider the following forgetful maps of equivalence classes of simply connected manifolds:

$$\left\{\begin{array}{l}\text{simply connected}\\ \text{smooth manifolds}\end{array}\right\} \xrightarrow{\Phi_1} \left\{\begin{array}{l}h\text{-cobordism}\\ \text{classes}\end{array}\right\} \xrightarrow{\Phi_2}$$
$$\left\{\begin{array}{l}\text{homotopy type} +\\ \text{pontryagin classes}\end{array}\right\} \xrightarrow{\Phi_3} \left\{\begin{array}{l}\text{cohomology ring} +\\ \text{pontryagin classes}\end{array}\right\}$$

The h-cobordism theorem show that Φ_1 is an isomorphism in dimension $n \geq 5$. Then surgery techniques and homotopy theory show that Φ_2, Φ_3 are finite-to-one, and furthermore the rational pontryagin classes are invariants of topological manifolds. This proves that any homeomorphism class of n-manifolds contains only finitely many diffeomorphism types, for $n \geq 5$. We will not comment further on this, as it would take us somewhat far afield.

In dimension 4, we can ignore the pontryagin classes, since by the Hirzebruch signature theorem, the only nontrivial pontryagin class p_1 encodes in the signature of the intersection form, which is in turn encoded by the cohomology ring. Amazingly, the map Φ_1 fails to be finite-to-one for 4-manifolds! But Φ_2 and Φ_3 are one-to-one. It is not difficult to prove this for Φ_3, using elementary methods from algebraic topology:

Theorem 12.8 (Whitehead)**.** *Let X and Y be compact, simply connected, smooth 4-manifolds. Then X and Y are homotopy-equivalent if and only if their intersection forms are isometric.*

Proof. Note that if X, Y are homotopy equivalent, they have isomorphic cohomology rings, and hence for an appropriate choice of orientation, the intersection forms are isometric. So the main content is the converse.

Let $X' = X - D^4$ be X minus a 4-ball. Mayer-Vietoris implies that $H_*(X', \mathbb{Z})$ is the same as $H_*(X, \mathbb{Z})$ except in degree 4, where it is zero. Additionally, X' is still simply connected. We have by the Hurewicz theorem that $\pi_2(X') = H_2(X')$ and thus, there is a map $\bigvee_r S^2 \to X'$ from a wedge of 2-spheres for which the fundamental class of the ith S^2 maps to the ith element e_i of a basis of $H_2(X')$. So $\bigvee_r S^2 \to X'$ is an isomorphism on integral homology, and thus is a weak homotopy equivalence (again by Hurewicz).

Since the source and target are both CW complexes, a well-known theorem in topology says that this map is a homotopy equivalence.

We conclude that X, and similarly Y, are homotopic to the gluing of a 4-ball D^4 along its boundary S^3 to a wedge of two spheres $\bigvee_r S^2$. It suffices to show that the intersection form encodes the homotopy class of the map $S^3 \to \bigvee_r S^2$ and two homotopic gluings will produce homotopy-equivalent topological spaces. Let $\gamma_1, \dots, \gamma_r$ be knots in S^3 with γ_i the inverse image of a generic point on the ith copy of S^2. Then

Exercise 12.9. *The linking number satisfies* $\mathrm{lk}(\gamma_i, \gamma_j) = e_i \cdot e_j$ *and the linking matrix is the unique homotopy invariant of the map* $S^3 \to \bigvee_r S^2$.

The theorem follows. □

Finally, we Freedman's best known theorem:

Theorem 12.10. *Two simply connected, closed, oriented topological* 4*-manifolds are homeomorphic if and only if (1) their intersection forms are isometric, and (2) they have equal Kirby-Siebenmann invariant (an invariant equal to* 0 *or* 1 *depending on if* $X \times \mathbb{R}$ *admits a smooth structure).*

Conversely, every even unimodular lattice is the second cohomology of some topological 4*-manifold. When the intersection form is odd, either Kirby-Siebenmann invariant can be realized, and when the intersection form is even, we necessarily have* $ks(X) \equiv \mathrm{sig}(X)/8$ *mod* 2*.*

The key to proving the second part is showing that there exists a *fake* 4*-ball* i.e. a topological 4-ball, bounding any homology 3-sphere. The key to proving the first part is to verify the topological h-cobodism theorem stated above, which relies on verifying a topological Whitney trick. This is very hard and we won't even begin to go into it.

Corollary 12.11. *There exist compact, oriented topological* 4*-manifolds with no smooth structure.*

Proof. By Freedman's theorem, there is a manifold M_{E_8} whose intersection form is the unique even unimodular lattice of signature $(0, 8)$. Its signature is $\mathrm{sig}(M_{E_8}) = -8$. If M_{E_8} had a smooth structure, it would be spin, because it is simply connected with even intersection form. But by Rokhlin's theorem the signature would be divisible by 16. Contradiction. □

Remark 12.12. Note that E_8 also appeared in the proof that there is no smooth 2-sphere representing the homology class $3h \in H^2(\mathbb{CP}^2, \mathbb{Z})$. The boundary of a tubular neighborhood of the strict transform $\widehat{\Sigma}$ of, say, a smooth Riemann surface Σ passing through the eight blown-up points is the famous *Poincare homology sphere.* By Freedman's theorem, it bounds a fake 4-ball and so the contraction exists in the topological category, and gives the E_8 manifold.

13. Classification of unimodular lattices

14. An exotic $\mathbb{R}^4$

15. The Seiberg-Witten equations