MATROIDS AND THE INTEGRAL HODGE CONJECTURE FOR ABELIAN VARIETIES

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Abstract. We prove that the cohomology class of any curve on a very general principally polarized abelian variety of dimension at least 4 is an even multiple of the minimal class. The same holds for the intermediate Jacobian of a very general cubic threefold. This disproves the integral Hodge conjecture for abelian varieties and shows that very general cubic threefolds are not stably rational. Our proof is motivated by tropical geometry; it relies on multivariable Mumford constructions, monodromy considerations, and the combinatorial theory of matroids.

Contents

1. Introduction

To a regular matroid $\underline{R}$ of rank g on an n element set, one may associate a degeneration $\pi\colon X \to B$ of principally polarized abelian varieties of dimension g over a n-dimensional base B. The goal of this paper is to connect combinatorial properties of $\underline{R}$ to algebro-geometric properties of the very general fiber of π—primarily, the existence of algebraic

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curves representing multiples of the minimal class. This leads to solutions to two long-standing open problems, one in the field of algebraic cycles on abelian varieties and one concerning rationality questions.

1.1. **Integral Hodge conjecture for abelian varieties.** By [AH62; Kol92], the integral Hodge conjecture fails in general. So its holding is a property of a variety. Understanding whether or not this property holds for a particular class of varieties is crucial for understanding their geometry.

An important case that has been open until now is the integral Hodge conjecture for abelian varieties. The specific case of curve classes on principally polarized abelian varieties is of particular interest; this question goes back (at least) to the work of Barton and Clemens [BC77, p. 66] and has been advertised for instance by Voisin [Voi17; Voi24], who found a relation to the stable rationality problem for rationally connected threefolds.

If (X, Θ) is a very general principally polarized abelian variety of dimension g, then the group of Hodge classes in $H^{2c}(X, \mathbb{Z})$ is generated by $[\Theta]^c/c!$, which is well-known to be a primitive integral cohomology class. Therefore, the integral Hodge conjecture for such X reduces to the question which multiples of $[\Theta]^c/c!$ are represented by a linear combination of closed subvarieties. Our result on this problem is as follows.

Theorem 1.1. *Let (X, Θ) be a very general principally polarized abelian variety of dimension $g \geq 4$. Let $Z \in \mathrm{CH}^c(X)$ be an algebraic cycle of codimension $2 \leq c \leq g-1$. Then $[Z] = m \cdot [\Theta]^c/c! \in H^{2c}(X, \mathbb{Z})$ with m even.*

The above theorem is equivalent to saying that for any cycle $Z \in \mathrm{CH}^c(X)$ of codimension $2 \leq c \leq g-1$, the intersection number $Z \cdot \Theta^{g-c}$ is divisible by $2 \cdot \frac{g!}{c!}$. For instance, Theorem 1.1 implies that for any curve $C \subset X$, the intersection number $C \cdot \Theta$ of C with Θ is divisible by $2g$. The proof of Theorem 1.1 quickly reduces to the case where $c = g-1$, i.e. to the case of curve classes that we will handle in this paper.

While the rational Hodge conjecture for abelian fourfolds has recently been proved by Markman [Mar25], the image of the integral cycle class map remains mysterious. Theorem 1.1 solves that problem for the very general principally polarized abelian fourfold:

Corollary 1.2. *Let (X, Θ) be a very general principally polarized abelian fourfold. Then*

$$\mathrm{im}(\mathrm{cl}^c \colon \mathrm{CH}^c(X) \to H^{2c}(X, \mathbb{Z})) = \begin{cases} 2\mathbb{Z}[\Theta]^c/c! & \text{if } c = 2, 3; \\ \mathbb{Z}[\Theta]^c/c! & \text{otherwise.} \end{cases}$$

1.2. **Cubic threefolds.** Our method is flexible and works for other families of abelian varieties, including the important case of intermediate Jacobians of cubic threefolds:

Theorem 1.3. *Let $Y \subset \mathbb{P}^4_{\mathbb{C}}$ be a very general cubic hypersurface. Then the homology class of any curve $C \subset JY$ on its intermediate Jacobian JY is an even multiple of the minimal class $[\Theta_Y]^4/4! \in H_2(JY, \mathbb{Z})$.*

By [Voi17], Theorem 1.3 implies:

Corollary 1.4. *Very general cubic threefolds $Y \subset \mathbb{P}^4_{\mathbb{C}}$ do not admit a decomposition of the diagonal. Hence they are neither stably rational, nor retract rational, nor $\mathbb{A}^1$-connected.*

The above result should be compared to the celebrated result of Clemens–Griffiths [CG72], who showed that for a smooth cubic threefold Y, the intermediate Jacobian (JY, Θ_Y) is not isomorphic to a product of Jacobians, which by Matsusaka's criterion is equivalent to saying that the minimal class $[\Theta_Y]^4/4!$ is not represented by the class of an effective algebraic curve on JY. This implied that Y is not rational, while the stronger conclusion that $[\Theta_Y]^4/4!$ is not algebraic for very general Y implies that such cubics are not stably rational. While every smooth cubic threefold is irrational, the smooth cubic threefolds which admit a decomposition of the diagonal form a (non-empty) countable union of subvarieties of their moduli, see [Voi17].

We remark that in contrast to the case treated in Theorem 1.1, the minimal surface class $[\Theta_Y]^3/3!$ is algebraic and in fact effective on JY, represented by the Abel–Jacobi image of the Fano surface of lines on Y, see [CG72].

In [BGF23], it has been shown that the integral Hodge conjecture holds for products of Jacobians of curves, and hence for abelian varieties that admit a split embedding into such a product. Voisin [Voi24] proved the converse, showing that in fact the integral Hodge conjecture for curve classes on abelian varieties is equivalent to the statement that any abelian variety admits a split embedding into a product of Jacobians of curves. Our results show that this fails in general:

Corollary 1.5. *Let (X, Θ) be a very general principally polarized abelian variety of dimension at least 4 or the intermediate Jacobian of a very general cubic threefold. Then, for any abelian variety Y, any isogeny $f\colon X \times Y \to \prod JC_i$ to a product of Jacobians of curves has even degree. In particular, X does not admit a split embedding into a product of Jacobians of curves.*

Note that any abelian variety X is, up to isogeny, a factor in the Jacobian of a curve, namely in the Jacobian of any curve $C \subset X$ which generates X as an abelian variety. The above corollary shows that in general, there are restrictions on the degree of the resulting isogeny $f\colon X \times Y \to JC$. Related results have been discussed in [GFS25].

1.3. **Algebraic curves on matroidal families.** In this section we explain a more precise result, which implies the results in Sections 1.1 and 1.2.

Let $\underline{R}$ be a regular matroid of rank g on a ground set S. To this data we may associate a smooth projective family $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ of g-dimensional principally polarized abelian varieties over the punctured polydisk $(\Delta^\star)^S$ with base point t, such that the associated vanishing cycles y_s, viewed via the polarization as linear forms on $\mathrm{gr}_0^W H_1(X_t, \mathbb{Z})$, realize the matroid $\underline{R}$. The monodromy about the s-th coordinate hyperplane is then encoded by some multiple of the quadratic form y_s^2, cf. Definition 2.12. We may further choose the family so that it extends to an algebraic family $\pi^\star\colon X^\star \to B^\star$ over a smooth quasi-projective base $B^\star$ with $(\Delta^\star)^S \subset B^\star$, see [EGFS25a, Propositions 4.10 and 5.11]. We call $\pi^\star$ a *matroidal family* associated to $\underline{R}$, see Definition 4.1 below.

Theorem 1.6. *Let $\underline{R}$ be a regular matroid of rank g. Let $\pi^\star\colon X^\star \to B^\star$ be a matroidal family of g-dimensional principally polarized abelian varieties associated to $\underline{R}$. Let $\iota\colon C_t \to X_t$ be a morphism from a projective curve into a very general fiber of π, with*

$$\iota_*[C_t] = m \cdot [\Theta]^{g-1}/(g-1)! \in H_2(X_t, \mathbb{Z}).$$

If $\underline{R}$ is not cographic, then m is even.

A matroid $M^*(G)$ is *cographic* if it can be realized by the natural map $E \to H_1(G, \mathbb{Z})^*$, where G is some oriented graph with edge set $E = E(G)$. On the other hand, the map $E \to \mathbb{Z}^E/H_1(G, \mathbb{Z})$ defines the *graphic* matroid $M(G)$. The graphic matroid of G is isomorphic to a cographic matroid if and only if G is planar. In addition to graphic and cographic matroids, there are regular matroids that are not related to graphs at all, such as the $\underline{R}_{10}$ matroid or any matroid with $\underline{R}_{10}$ as a minor, cf. [Oxl92, Corollary 13.2.5].

Our obstruction is topological in the sense that the matroidal information is encoded in the monodromy, which in turn depends only on the topological type of the family $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ over the punctured polydisc, that is, only on the homeomorphism type of the corresponding real torus bundle over $(S^1)^S$. Cographic matroids are precisely those that appear as the vanishing cycles of a degeneration of curves. In fact, $\underline{R}$ is cographic if and only if $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ deforms to a family of Jacobians of curves, see [EGFS25a, Remark 2.31]. The above theorem thus shows that for a matroidal family of principally polarized abelian varieties, the integral Hodge conjecture fails for curve classes on very general fibers, unless the family is deformation equivalent over $(\Delta^\star)^S$ to a family of Jacobians, see Lemma 2.8 and Corollary 8.2 below.

Theorem 1.1 for curve classes on abelian fourfolds will follow by applying Theorem 1.6 to the graphic matroid $M(K_5)$ associated to the complete graph K_5, which is not planar, hence the graphic matroid $M(K_5)$ of rank 4 is not cographic. Similarly, Theorem 1.3 follows by applying Theorem 1.6 to a matroidal family of intermediate Jacobians of cubic threefolds associated to $\underline{R}_{10}$ due to Gwena [Gwe04], see Section 8 below.

1.4. **Obstruction and combinatorial results.** We sketch some ideas behind Theorem 1.6. Replacing C_t by its normalization, we may assume that C_t is smooth projective. We then get a morphism $f\colon JC_t \to X_t$. Using the principal polarizations on both sides, the dual of f defines a map $f^\vee : X_t \to JC_t$. Since $f_*[C_t] = m \cdot [\Theta]^{g-1}/(g-1)!$, we find that $(f^\vee)^*[\Theta_{C_t}] = m \cdot [\Theta]$ and the composition $f \circ f^\vee\colon X_t \to X_t$ is multiplication by m.

Let Λ be a ring in which m is invertible. Then the above observations show that $\frac{1}{m} f_*^\vee$ splits the map f_* and we get a canonical decomposition

$$H_1(C_t, \Lambda) \cong H_1(X_t, \Lambda) \oplus \ker(f_*\colon H_1(C_t, \Lambda) \to H_1(X_t, \Lambda)). \tag{1.4.1}$$

One checks that this decomposition is orthogonal with respect to Θ_{C_t} (Lemma 2.10). Since $t \in B^\star$ is very general, there exists a generically finite cover $B'^\star \to B^\star$ such that we may spread out the curve C_t over an open subset of $B'^\star$. To explain the mechanism of our proof, we will ignore this (important) technical difficulty and assume that $B'^\star = B^\star$, i.e. that the above decomposition holds over a Zariski open subset of $B^\star$, and hence is respected by all monodromy operators.

Let us now fix $t \in (\Delta^\star)^S \subset B^\star$. Then the decomposition (1.4.1) is respected by the monodromy operator T_s about the s-th coordinate hyperplane of Δ^S. Assume moreover that T_s is unipotent with nilpotent logarithm N_s and note that the decomposition (1.4.1) is compatible with the monodromy weight filtration. Taking gr_0^W, we obtain a decomposition

$$\mathrm{gr}_0^W H_1(C_t) \cong \mathrm{gr}_0^W H_1(X_t) \oplus \ker(f_*\colon \mathrm{gr}_0^W H_1(C_t) \to \mathrm{gr}_0^W H_1(X_t)),$$

where homology is taken with Λ coefficients, and where we assume $\Lambda \subset \mathbb{R}$ for simplicity. We may then define the monodromy bilinear form Q_s on $\mathrm{gr}_0^W H_1(C_t)$ by the formula $Q_s(u, v) \coloneqq \Theta_{C_t}(N_s u, v)$, see Definition 2.12 below. The above decomposition is orthogonal with respect to Q_s and the restriction of Q_s to $\mathrm{gr}_0^W H_1(X_t, \Lambda)$ agrees with m times the monodromy bilinear form $B_s(u, v) = \Theta(N_s u, v)$. By the assumption that $\pi^\star\colon X^\star \to B^\star$ is matroidal, these monodromies are given by a multiple of the bilinear form $u \otimes v \mapsto y_s(u) y_s(v)$, which we denote by y_s^2.

Let us, in addition, assume that the curve C_t has a unique nodal limit C_0 over the origin of Δ^S. Then $\mathrm{gr}_0^W H_1(C_t, \mathbb{Z}) \cong H_1(\Gamma(C_0), \mathbb{Z})$ identifies to the homology of the dual complex of C_0, cf. [EGFS25a, Proposition 5.10]. Under all of the above assumptions, we then see that $(\underline{R}, S)$ admits a Λ-splitting of level $d = m$ in the cographic matroid $M^*(G)$ of some graph G, in the following sense (cf. Section 2.5 below):

Definition 1.7. Let $(\underline{R}, S)$ and $(\underline{M}, E)$ be regular matroids with integral realizations $S \to U^*$, $s \mapsto y_s$, and $E \to V^*$, $e \mapsto x_e$, respectively. Let Λ be a ring and let d be a positive integer. A *quadratic Λ-splitting of level d* of $(\underline{R}, S)$ in $(\underline{M}, E)$ consists of an embedding $U_\Lambda \hookrightarrow V_\Lambda$ and a decomposition

$$V_\Lambda = U_\Lambda \oplus U' \tag{1.4.2}$$

for some $U' \subset V_\Lambda$, together with a matrix $(a_{se}) \in \mathbb{Z}_{\geq 0}^{S\times E}$ of non-negative integers, such that, for all $s \in S$, the bilinear form $Q_s := \sum_e a_{se} x_e^2$ has the following properties:

(1) the decomposition (1.4.2) is orthogonal with respect to Q_s;
(2) the restriction of Q_s to U_Λ agrees with $d \cdot y_s^2$.

The above definition is key to our proof of Theorem 1.6, which falls naturally into two parts: the reduction to combinatorics, and the proof of the resulting combinatorial statement. The reduction to combinatorics is achieved by the following algebro-geometric result, which is proven without any of the aforementioned simplifying assumptions.

Theorem 1.8. *Let $\pi^\star \colon X^\star \to B^\star$ be a matroidal family of principally polarized abelian varieties associated to a regular matroid $(\underline{R}, S)$. Let $\iota \colon C_t \to X_t$ be a morphism from a projective curve into a very general fiber of $\pi^\star$, with*

$$\iota_*[C_t] = m \cdot [\Theta]^{g-1}/(g-1)! \in H_2(X_t, \mathbb{Z}).$$

Let ℓ be a prime that is coprime to m. Then there is a positive integer d such that $(\underline{R}, S)$ admits a quadratic $\mathbb{Z}_{(\ell)}$-splitting of level d in a cographic matroid.

The main combinatorial result that we prove and which allows us to finish the proof of Theorem 1.6 is then the following:

Theorem 1.9. *Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$. Then $(\underline{R}, S)$ admits a quadratic $\mathbb{Z}_{(2)}$-splitting of some level $d \geq 1$ in a cographic matroid if and only if $(\underline{R}, S)$ is cographic.*

In contrast to the above result, there are many non-cographic matroids, such as $M(K_5)$, $M(K_{3,3})$, and $\underline{R}_{10}$, that admit quadratic $\mathbb{Z}_{(\ell)}$-splittings in cographic matroids for all odd primes ℓ, see Remark 8.6 below. Moreover, we have the following general result.

Theorem 1.10. *Let $(\underline{R}, S)$ be a regular matroid of rank g and with integral realization $S \to U^*$. If $\ell \geq g$ is a prime, then $(\underline{R}, S)$ admits a quadratic $\mathbb{Z}_{(\ell)}$-splitting of some level d in a cographic matroid.*

1.5. **Outline.** As aforementioned, our main results follow from Theorem 1.6, which itself reduces to Theorems 1.8 and 1.9: To prove Theorem 1.6, assume contrapositively that m is odd. Then Theorem 1.8 implies that there is a $\mathbb{Z}_{(2)}$-splitting of $\underline{R}$ of some level d in a cographic matroid. Hence, by Theorem 1.9, $\underline{R}$ is cographic.

1.5.1. *Outline of the proof of Theorem 1.8.* In the set-up of Theorem 1.8, we choose a smooth partial compactification $B^\star \subset B$ such that $H = B \setminus B^\star$ is an snc divisor on B, $(\Delta^\star)^S \subset B^\star$ extends to an embedding $\Delta^S \subset B$, and $H \cap \Delta^S$ is given by the vanishing of the coordinate functions. We further perform the necessary base change $\tau\colon B' \to B$ over which the aforementioned curve C_t spreads out. Here we want B' to be regular and the preimage of the coordinate axes to be snc (approaches with singular B' and finite τ were unsuccessful). In particular, τ will not be finite in general. In fact, $\tau^{-1}(0)$ can be assumed to be an snc divisor, which implies that we do not have a single limit curve C_0 at our disposal anymore. This poses serious technical difficulties. For instance, using the universal family over $\overline{\mathcal{M}}_g$, we may after further blow-up assume that C_t extends to a nodal family over B'. But this only allows one to deduce a "patch" version of the notion in Definition 1.7—that is, a weaker version corresponding to patching together the monodromy information we obtain at the deepest strata of $\tau^{-1}(0)$. However, there are examples that show that this weaker combinatorial property is not enough to conclude what we want and that, moreover, the regularity of $\underline{R}$ has to play an important role. The crucial idea to circumvent this issue is to first control the limits of the abelian varieties and then to produce a family of curves in such a way that the limits of the curves are closely related to the limits of the abelian varieties. This provides enough control of the combinatorics of the singular curves, allowing us to deduce Theorem 1.8.

The key starting input, exploiting the regularity of $\underline{R}$, is [EGFS25a, Theorem 7.1], which produces a regular, flat extension $\pi\colon X \to B$ of the matroidal family $\pi^\star\colon X^\star \to B^\star$, such that π is H*-nodal*, see Definition 2.4. We perform a detailed analysis of a specific resolution $X'' \to X' = X \times_B B'$ which is semistable over B' in Section 3. In Section 4, we replace C_t by a particularly nice representative of some ℓ-prime multiple of the minimal class which is a complete intersection in some projective bundle over X''. We then use the resolution $X'' \to X'$ to construct a graph G via a careful gluing procedure of pieces of the dual graphs of the limits of the curve C_t over certain snc strata of $\tau^{-1}(0)$.

The construction of G begins by relating the dual graph of each limiting nodal curve to the 1-skeleton of the dual complex of the corresponding fiber of $X'' \to B'$. By the explicit resolution $X'' \to X'$, we relate that 1-skeleton to the 1-skeleton of the dual complex of the respective fiber of $X' \to B'$; the latter is constant along $\tau^{-1}(0)$, because $X' = X \times_B B'$ is a fiber product. This allows us to construct G via "edge multiplication" from the 1-skeleton $\Gamma^1(X_0)$ of the dual complex of X_0. Our analysis shows that $\underline{R}$ admits a quadratic $\mathbb{Z}_{(\ell)}$-splitting of some level d into the cographic matroid $M^*(G)$.

1.5.2. *Outline of the proof of Theorem 1.9.* In Section 5, we show that a quadratic $\mathbb{Z}_{(\ell)}$-splitting of level d of $(\underline{R}, S)$ in the cographic matroid $M^*(G)$ yields $\mathbb{Z}_{(\ell)}$*-solutions* in G; these are certain linear combinations of edges of G satisfying linear conditions dictated by $\underline{R}$. We then define Albanese graphs $\mathrm{Alb}_{\ell^r,\ell^j} := \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ associated to $(\underline{R}, S)$. These Albanese graphs satisfy a universal property, yielding a combinatorial Albanese map $\mathrm{alb}\colon G \to \mathrm{Alb}_{\ell^r,\ell^j}$ along which we can push forward a given solution. In this way, we also obtain $\mathbb{Z}_{(\ell)}$-solutions in the graphs $\mathrm{Alb}_{\ell^r,\ell^j}$. Applying $\otimes_{\mathbb{Z}_{(\ell)}}\Lambda$, we obtain such solutions for any $\mathbb{Z}_{(\ell)}$-algebra Λ.

In Section 6, we study divisibility of solutions to go from ℓ^i-indivisible Λ-solutions in $\mathrm{Alb}_{\ell^r,\ell^j}$ to ℓ^{i-j}-indivisible Λ-solutions in $\mathrm{Alb}_{\ell^{r-j},1}$. Applying this to the case $r = j+1$ and $\ell = 2$, it then suffices to show that the existence of Λ-solutions in $\mathrm{Alb}_{2,1}$ implies that $\underline{R}$ is cographic. In Section 7, we show that the property of having an ℓ^i-indivisible Λ-solution in $\mathrm{Alb}_{\ell^r,\ell^j}$ is closed under taking minors. By a theorem of Tutte, the class of cographic matroids can be described as those regular matroids that do not have $M(K_5)$ and $M(K_{3,3})$ as minors. We will use this to reduce Theorem 1.9 to showing that those two excluded minors do not have 2-indivisible $\mathbb{Z}/2$-solutions in $\mathrm{Alb}_{2,1}$. This in turn reduces to an explicit rank computation of certain matrices, see Proposition 7.5.

Remark 1.11. Recall that the integers a_{se} in Definition 1.7 are non-negative; Theorem 1.9 fails if one drops that assumption. This explains the choice of coefficients $\mathbb{Z}_{(\ell)}$ (opposed to $\mathbb{Z}_\ell$, say), because $\mathbb{Z}_{(\ell)} \subset \mathbb{R}$ and we will exploit this in the proof, see Theorem 5.10 and Lemma 5.12 below.

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2. Preliminaries

2.1. Conventions.

2.1.1. *Schemes and analytic spaces.* We work, if not mentioned otherwise, over the field of complex numbers. We frequently identify a complex algebraic scheme or variety with the corresponding complex analytic space. If $Y \to B$ and $B' \to B$ are morphisms of complex analytic spaces, we denote the corresponding base change by $Y_{B'} := Y \times_B B'$. We denote by $\Delta \subset \mathbb{C}$ the open unit disc.

An snc divisor D on a smooth complex analytic space X is a divisor which is locally isomorphic to the union of some coordinate hyperplanes in Δ^n. If D_i with $i \in I$ denote the components of D, then the strata of D are the intersections $D_J := \bigcap_{j\in J} D_j$ for $J \subset I$; the respective open stratum is given by $D_J^\circ = D_J \setminus \bigcup_{i\in I\setminus J} D_i$.

2.1.2. *Linear algebra.* If U is a free $\mathbb{Z}$-module and Λ is a ring, then $U_\Lambda := U \otimes_{\mathbb{Z}} \Lambda$ denotes the tensor product.

Let ℓ be a prime and let x be an element of an abelian group. An *ℓ-prime multiple* of x is a multiple $m \cdot x$ such that $m \in \mathbb{Z}$ is coprime to ℓ. For instance, 2-prime multiples are nothing but odd multiples of x.

Let V be a free $\mathbb{Z}$-module of finite rank and let $l_i \in V^* = \mathrm{Hom}(V, \mathbb{Z})$ be linear forms. Then we will often denote the bilinear form $(x, y) \mapsto \sum a_i l_i(x) l_i(y)$ on V by $Q = \sum a_i l_i^2$ and identify it with the attached quadratic form.

2.1.3. *Graphs.* All graphs are finite. An *orientation* of a graph G is an orientation of its edges. If not mentioned otherwise, then the chain complex $C_*(G, \Lambda)$ and the homology $H_*(G, \Lambda)$ of a graph G with coefficients in some ring Λ refers to the simplicial chain complex, resp. simplicial homology.

Let S be a set. A *partial S-coloring* of a graph G is a partition $\sqcup_{s\in S} E_s \subset E$ of a subset of the edges of E. Edges in E_s are called edges of *color* s; edges in $E \setminus \sqcup_{s\in S} E_S$ are *colorless* edges. An *S-coloring* of a graph G is a partial S-coloring without colorless edges, i.e. with $\sqcup_{s\in S} E_S = E$. A *morphism* of (partially) S-colored graphs is a map of graphs that respects colors, i.e. it maps edges of color s to edges of color s and it maps colorless edges to colorless edges. An *(I, S)-bicolored* graph is a graph G which carries colorings with respect to two sets I and S. A *morphism* of (I, S)-bicolored graphs is a morphism which respects both colorings.

Let G be an S-colored graph. We say that a 1-chain $\alpha \in C_1(G, \Lambda)$ has *color s* if it is a linear combination of (oriented) edges of color s. Any 1-chain $\alpha \in C_1(G, \Lambda)$ can uniquely be written as a sum $\alpha = \sum_{s\in S} \alpha_s$ of 1-chains α_s of color s. The 1-chain α_s will be referred to as the *s-colored part* of α.

2.1.4. *Cell complexes.* We frequently identify a (finite) cell or polyhedral complex of dimension 1 with the associated graph and vice versa. A *refinement* of a graph G is a refinement of the associated cell complex, i.e. it is the graph obtained from G by replacing certain edges by chains of edges.

Let $X := Y_p$ be a fiber of D-semistable or a D-quasi-nodal morphism $Y \to B$, see Definitions 2.4 and 3.1 below. In the former case, X analytically-locally has, by the local form (3) in Definition 2.4, product-of-snc singularities $\prod_{i\in I'} \{x_i^{(1)} \cdots x_i^{(m_i)} = 0\}$. In the latter case, X has singularities which are a product of nodal singularities $\{x_s y_s = 0\}$ with a smooth variety.

Let $X^\nu := X^{[0]}$ be the normalization, and more generally, $X^{[k]}$ be the normalization of all codimension k Whitney strata of X. If X is a fiber of a strict D-semistable morphism (cf. Remark 2.5), we may write $X = \bigcup_{j\in J} X_j$ as a union of smooth components, with smooth intersections $X_{J'} := \bigcap_{j\in J'} X_j$ for $J' \subset J$. We refer to the irreducible components of these intersections as the *strata* of X and they form the irreducible components of $X^{[k]}$, though note that in general, $k \neq |J|$.

The *dual complex* $\Gamma(X)$ is a polyhedral complex whose k-dimensional polyhedral cells are in bijection with the components of $X^{[k]}$, and which are glued according to the incidences of strata. For example, for the snc variety $x^{(1)} \cdots x^{(m)} = 0$, the dual complex is an $(m-1)$-simplex, and for the product of snc varieties $\prod_{i\in I'}\{x_i^{(1)} \cdots x_i^{(m_i)} = 0\}$, the dual complex is a product of $(m_i - 1)$-simplices, with its natural product polyhedral structure. Thus, in the case of a D-quasi-nodal morphism, the dual complex is a gluing of *cubes*, i.e. products of 1-simplices, along various faces.

We denote the *k-skeleton* $\Gamma^k(X) \subset \Gamma(X)$ of $\Gamma(X)$ as the union of all polyhedral faces of dimension $\leq k$. In particular, $\Gamma^1(X) \subset \Gamma(X)$ is a graph, encoding the incidences of the irreducible components and double loci of X. We observe that the above constructions work more generally, for any toroidal morphism $Y \to B$; but, in general, the polyhedral cells are more complicated (and we will not need this).

2.1.5. *Matroids.* A matroid $\underline{R}$ is a pair consisting of a finite set S, called the ground set of $\underline{R}$, and a subset $\mathcal{I} \subset \mathcal{P}(S)$ of its power set, whose elements are called the independent sets of $\underline{R}$, satisfying $\emptyset \in \mathcal{I}$, the downward-closed property, and the independent set exchange property, see [Oxl92]. If we want to highlight the ground set, we will also write $(\underline{R}, S)$ instead of $\underline{R}$. An *integral realization* is a map $S \to V$ to a free $\mathbb{Z}$-module V, which induces a realization of $\underline{R}$ over any field k, i.e. over V_k for k arbitrary. Equivalently, any subset of S generates a saturated sublattice of V. In particular, a matroid with an integral realization is *regular*—it admits a realization over any field. Conversely, every regular matroid admits an integral realization by a *totally unimodular matrix*, i.e. a matrix all of whose minors have determinant $\{0, \pm 1\}$, see [Oxl92, Theorem 6.6.3].

We always assume that the rank of $\underline{R}$ agrees with the rank of V. In other words, the image of S generates V as a $\mathbb{Z}$-module; this implies that the dual map $V^* \to \mathbb{Z}^S$ is injective. In this paper, the free $\mathbb{Z}$-module V frequently occurs as the dual of another free $\mathbb{Z}$-module.

2.2. Integral realizations and loopless matroids.

Lemma 2.1. *Let $(\underline{R}, S)$ be a regular matroid of rank g with integral realization $S \to V$. Let $B \subset S$ be a basis of $\underline{R}$. Then the matrix $(\mathbb{1}_g | D)$ that represents the linear map $\mathbb{Z}^S \to V$ in the basis of V defined by B is totally unimodular.*

Proof. Since $S \to V$ is an integral realization, any subset of S generates a saturated sublattice of V. It follows that all maximal minors are contained in $\{0, \pm 1\}$. Let now M be a non-maximal square submatrix of $(\mathbb{1}_g|D)$. We may assume that M does not contain a zero column. Then, we may extend M to a maximal square submatrix, by adding rows and columns corresponding to an identity submatrix of the $\mathbb{1}_g$ block. This leaves the determinant of M the same up to sign, and so reduces to the earlier case. $\square$

We discuss now uniqueness of integral realizations.

Lemma 2.2. *Let $(\underline{R}, S)$ be a regular matroid with integral realizations $\varphi_1 : S \to V_1$ and $\varphi_2 : S \to V_2$. Let $g = \operatorname{rk} V_i$ be the rank of $\underline{R}$. Then there is a commutative diagram*

$$\begin{tikzcd} \mathbb{Z}^S \arrow[r] \arrow[d, "\cong"] & V_1 \arrow[d, "\cong"] \\ \mathbb{Z}^S \arrow[r] & V_2 \end{tikzcd}$$

where the vertical maps are isomorphisms of $\mathbb{Z}$-modules, and the horizontal maps are the linear maps induced by φ_1 and φ_2, respectively. More precisely, if we choose a basis $B \subset S$ of $\underline{R}$ and consider the associated matrix $(\mathbb{1}_g|D_i) \in \mathbb{Z}^{g\times S}$ that represents $\mathbb{Z}^S \to V_i$ in the given basis, then $(\mathbb{1}_g|D_2)$ can be obtained from $(\mathbb{1}_g|D_1)$ by multiplying some rows and columns by -1.

Proof. By Lemma 2.1, the choice of a basis $B \subset S$ of $\underline{R}$ allows us to represent the linear maps $\mathbb{Z}^S \to V_1$ and $\mathbb{Z}^S \to V_2$, induced by φ_1 and φ_2, by totally unimodular matrices $(\mathbb{1}_g|D_1)$ and $(\mathbb{1}_g|D_2)$ for $D_i \in \mathbb{Z}^{g\times(S\setminus B)}$, respectively. By [Oxl92, Proposition 6.4.1], $D_1^\sharp = D_2^\sharp$, where $D_i^\sharp$ is the matrix whose entries are the absolute values of the respective entries of D_i. By [Oxl92, Lemma 13.1.6], D_2 can thus be obtained from D_1 by multiplying some rows and columns by -1. This implies that we can obtain $(\mathbb{1}_g|D_2)$ from $(\mathbb{1}_g|D_1)$ by multiplying some rows and columns by -1. $\square$

We say that $(\underline{R}, S)$ is *loopless* if every singleton is an independent set. If $(\underline{R}, S)$ is regular with integral realization $S \to V$, then the associated totally unimodular matrix M has no zero column if and only if $(\underline{R}, S)$ is loopless.

Lemma 2.3. *If $S \to V$ is an integral realization of a loopless regular matroid $(\underline{R}, S)$, then for each $s \in S$, the composition $V^* \to \mathbb{Z}^S \xrightarrow{\operatorname{pr}_s} \mathbb{Z}$ is surjective, where $\operatorname{pr}_s \colon \mathbb{Z}^S \to \mathbb{Z}$ denotes the projection to the s-th coordinate.*

Proof. By Lemma 2.1, $\mathbb{Z}^S \to V$ may be represented by a totally unimodular matrix $M \in \mathbb{Z}^{g\times S}$. Since $(\underline{R}, S)$ is loopless, M has no column which is identically zero. Hence, the transpose M^t, which represents $V^* \to \mathbb{Z}^S$, has the property that each row contains an element ± 1. The lemma follows from this property. $\square$

2.3. **D-nodal, nearly D-nodal, and D-semistable morphisms.** We use the following terminology, see also [EGFS25a, Definition 5.1].

Definition 2.4. Let (B, D) be a pair of a smooth complex analytic space B and an snc divisor D with components D_i, $i \in I$. We say that a morphism $f\colon Y \to B$ of complex analytic spaces is

(1) *D-nodal* if locally in the analytic topology, f is of the form
$$\textstyle\prod_{i\in I'}\{u_i = x_i y_i\} \times \Delta^{j+k} \to \prod_{i\in I'} \Delta_{u_i} \times \Delta^j,$$
where $I' \subset I$, u_i is a local equation for D_i, and $\Delta^{j+k} \to \Delta^j$ is the projection to the first j coordinates;

(2) *nearly D-nodal* if we rather have a normal form of shape
$$\textstyle\prod_{i\in I'}\{u_i = x_i^{(1)} y_i^{(1)} = \cdots = x_i^{(m_i)} y_i^{(m_i)}\} \times \Delta^{j+k} \to \prod_{i\in I'} \Delta_{u_i} \times \Delta^j;$$

(3) *D-semistable* if we have
$$\textstyle\prod_{i\in I'}\{u_i = x_i^{(1)} x_i^{(2)} \cdots x_i^{(m_i)}\} \times \Delta^{j+k} \to \prod_{i\in I'} \Delta_{u_i} \times \Delta^j.$$

If moreover the generic fiber of $Y \times_B D_i \to D_i$ has smooth components for all $i \in I$, then f is called *strict* D-nodal, *strict* nearly D-nodal, *strict* D-semistable, respectively.

Remark 2.5. The strictness condition, together with the given local normal forms, ensure that the components of $Y \times_B D_i$ are regular and have, in particular, no self-intersections.

Definition 2.4 works equally well in the algebraic category, by requiring the given local forms, in étale charts. Next, we prove a Bertini-type theorem for hyperplane sections of strict D-semistable morphisms. We work in the algebraic category, as it makes the existence of a certain Zariski open subset slightly more transparent.

Lemma 2.6. *Let B be a smooth quasi-projective variety over an infinite field with snc divisor $D \subset B$ and let $Y \to B$ be a projective strict D-semistable morphism of relative dimension $g \geq 1$. Let $\mathcal{E} = L_1 \oplus \cdots \oplus L_c$ be a sum of very ample line bundles L_i on Y, $c \leq g$, and let $P \subset B$ be a finite set of points. Let $Z \subset Y$ be the zero locus of a general section of $\mathcal{E}$.*

Then there is a Zariski open subset $B^\circ \subset B$ which contains P, such that:

(1) *The base change $Z_{B^\circ} \to B^\circ$ is strict D°-semistable, where $D^\circ := B^\circ \cap D$.*
(2) *The local normal form of $Z_{B^\circ} \to B^\circ$ at a point $z \in Z_{B^\circ}$ is the same as that of $Y_{B^\circ} \to B^\circ$ at z, up to reducing the value of k in item (3) of Definition 2.4.*
(3) *For $p \in B^\circ$, the positive-dimensional strata of Z_p are in bijective correspondence with the strata of dimension $\geq c+1$ of Y_p.*

Proof. Let Y_p be the fiber of $Y \to B$ at a point $p \in P$. As the ground field is infinite, Bertini's theorem implies that we may ensure that

(1) Z is smooth of codimension c, and
(2) for each $p \in P$, Z intersects all strata of Y_p transversely.

In particular, Z is disjoint from all strata of Y_p of dimension $\leq c-1$, intersects each stratum of dimension c at a finite set of points, and intersects each stratum of dimension $k \geq c+1$ in a smooth irreducible $(k-c)$-dimensional subvariety of that stratum.

Let $z \in Z_p$ and suppose that the étale local form of the morphism $Y \to B$ at z is given by $W \times \mathbb{A}^{j+k} \to \prod_{i \in I'} \mathbb{A}_{u_i} \times \mathbb{A}^j$, where $W := \prod_{i \in I'}\{u_i = x_i^{(1)} \cdots x_i^{(m_i)}\}$, cf. item (3) in Definition 2.4. We may assume that z corresponds to the origin in this chart. That is, there is a Zariski open subset $U \subset Y$ with $z \in U$ and an étale map $U \to W \times \mathbb{A}^{j+k}$ which sends z to $(0,0)$ and which is compatible with the respective projections to B.

The transversality of the intersection, see (2) above, then implies $k \geq c$. We may thus consider the composition

$$Z \cap U \hookrightarrow U \longrightarrow W \times \mathbb{A}^j \times \mathbb{A}^k \longrightarrow W \times \mathbb{A}^j \times \mathbb{A}^{k-c}, \tag{2.3.1}$$

where the second arrow is induced by the identity on the first two factors and by the projection $\mathbb{A}^k \to \mathbb{A}^{k-c}$ to the first $k-c$ variables on the last factor. The above map is compatible with the respective projections to B. The lowest-dimensional strata of U_p corresponds in the above chart to $\{0\} \times \{0\} \times \mathbb{A}^k$. The fact that Z intersects this stratum transversely thus implies that, up to a linear change of coordinates on $\mathbb{A}^k$, we may assume that the composition in (2.3.1) induces an isomorphism on tangent spaces at z. Hence, up to shrinking U, we may assume that (2.3.1) is étale. We have thus produced a Zariski open subset $U \subset Y$ with $z \in U$ such that items (1)–(3) hold on U.

Since P is finite, we are able to find a finite number of étale charts as above that cover Y_p for all $p \in P$. From this one easily derives the existence of a Zariski open subset $B^\circ \subset B$ that has the properties in the lemma. This concludes the proof. ☐

Remark 2.7. In this paper, we will apply the above lemma in the case $c = g-1$, so that $Z_{B^\circ} \to B^\circ$ is a family of nodal curves, because the only local normal form in (3) giving a codimension 1 singular stratum of a fiber is $u_i = x_i y_i$.

2.4. **Curves whose cohomology class is a multiple of the minimal class.** In the following lemma, we reduce the task of proving that an ℓ-prime multiple of the minimal class is not algebraic to showing that no ℓ-prime multiple is represented by an effective curve, which may in fact be assumed to be smooth and projective.

Lemma 2.8. *Let (X,Θ) be a principally polarized abelian variety of dimension $g \geq 3$. Let ℓ be a prime and assume that an ℓ-prime multiple of $[\Theta]^{g-1}/(g-1)!$ is represented* *by a $\mathbb{Z}$-linear combination of classes of algebraic curves. Then some (possibly different) ℓ-prime multiple of $[\Theta]^{g-1}/(g-1)!$ is represented by a smooth projective connected curve.*

Proof. By assumption, there is an integer m that is coprime to ℓ such that

$$m[\Theta]^{g-1}/(g-1)! = \sum_i a_i[C_i] \in H_2(X, \mathbb{Z}),$$

for some curves $C_i \subset X$ and integers $a_i \in \mathbb{Z}$. By a result of Hironaka [Hir68], we may assume that C_i is smooth for all i. Replacing C_i by a generic translation, we may further assume that the union $C \coloneqq \bigcup_i C_i$ is smooth (because $g \geq 3$). A general complete intersection surface $Y \subset X$ of elements in $|k\Theta|$ for $k \gg 0$ which contain C is then smooth by Bertini's theorem. For a sufficiently large integer b, the line bundle

$$\sum a_i C_i + \ell^b \Theta|_Y \in \operatorname{Pic} Y$$

on Y is globally generated by Serre's theorem, and hence represented by a smooth projective connected curve $C' \subset Y$. When viewed as a curve on X, we then have

$$[C'] = \sum a_i[C_i] + \ell^b[\Theta] \cdot [Y] = m[\Theta]^{g-1}/(g-1)! + \ell^b \cdot k^{g-2}[\Theta]^{g-1} \in H_2(X, \mathbb{Z}),$$

which is an ℓ-prime multiple of the minimal class, as we want. □

In the next lemma we work over an arbitrary field k and consider the ℓ-adic étale cohomology of varieties over the algebraic closure $\bar{k}$; this will later be applied to the generic fiber of a family of complex projective varieties.

Lemma 2.9. *Let X be an abelian variety over a field k, principally polarized by $\Theta \in \mathrm{NS}(X_{\bar{k}})$. Let ℓ be a prime number different from the characteristic of k. Let C be a smooth projective curve, possibly disconnected, together with a morphism $f\colon JC \to X$. Let m be a positive integer and consider the following properties.*

1. *(1) We have $f_*[C] = m \cdot \Theta^{g-1}/(g-1)! \in H^{2g-2}(X_{\bar{k}}, \mathbb{Z}_\ell(g-1))$.*
2. *(2) The dual map $f^\vee\colon X \to JC$ has the property that $(f^\vee)^*\Theta_C = m \cdot \Theta$.*
3. *(3) The composition $X \xrightarrow{f^\vee} JC \xrightarrow{f} X$ is multiplication by m.*
4. *(4) There is a vector bundle $\mathcal{E}$ on X such that $s_i(\mathcal{E}) = m^i \cdot \Theta^i/i!$ for each $i \geq 0$, where $s_i(\mathcal{E})$ denotes the i-th Segre class.*

Then (1) and (2) are equivalent, and imply (3) and (4).

Proof. Let $\lambda_X\colon X \xrightarrow{\sim} X^\vee$ (resp. $\lambda_C\colon JC \xrightarrow{\sim} JC^\vee$) be the symmetric isomorphism attached to the principal polarization Θ (resp. Θ_C). Let $f^\vee\colon X^\vee \to JC^\vee$ be the dual of f, and also denote by $f^\vee\colon X \to JC$ the composition $X \to X^\vee \to JC^\vee \to JC$ given by $\lambda_C^{-1} \circ f^\vee \circ \lambda_X$.

The equivalence of (1) and (2) is well-known (sometimes referred to as Welters' criterion); it follows from the fact that the chain of isomorphisms $H^2(X_{\bar{k}}, \mathbb{Z}_\ell(1)) \cong$

$H^{2g-2}(X_{\bar{k}}^\vee, \mathbb{Z}_\ell(g-1)) \cong H^{2g-2}(X_{\bar{k}}, \mathbb{Z}_\ell(g-1))$, maps Θ to $\Theta^{g-1}/(g-1)!$. The first isomorphism is given by Poincaré duality and the second isomorphism is induced by λ_X.

To prove (2)⇒(3), recall that the pull back $(f^\vee)^*(\lambda_C)$ is by definition the symmetric isogeny $X \to X^\vee$ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "f^\vee"] \arrow[d, "(f^\vee)^*(\lambda_C)"'] & JC \arrow[d, "\lambda_C"] \\ X^\vee & JC^\vee \arrow[l, "f"'] \end{tikzcd}$$

commute. As $(f^\vee)^*(\Theta_C) = m \cdot \Theta$ by (2), we get $(f^\vee)^*(\lambda_C) = m \cdot \lambda_X$. Hence the composition

$$X \xrightarrow{f^\vee} JC \xrightarrow{\lambda_C} JC^\vee \xrightarrow{f} X^\vee \xrightarrow{\lambda_X^{-1}} X$$

is multiplication by m. Hence, (2)⇒(3), as claimed.

It remains to prove (2)⇒(4). For an integer $n \geq 2g-1$, let P_n be a Poincaré bundle on $C \times J^nC$, where J^nC is the JC-torsor of isomorphism classes of line bundles of degree n. Push P_n forward to get a sheaf on J^nC, which is a vector bundle because $n \geq 2g-1$ (which implies that the higher cohomology of P_n on the fibers of $C \times J^nC \to J^nC$ vanishes). Pull this vector bundle back along the isomorphism $JC \cong J^nC$, defined by some degree n divisor on C, to get a vector bundle F on JC that satisfies $\mathrm{ch}(F) = \mathrm{rk}(F) - \Theta$ and $c_i(F) = (-1)^i\Theta_C^i/i!$ for all $i \geq 0$, see [Arb+85, p. 336]. In other words, the total Chern class is given by $c(F) = e^{-\Theta_C}$ and hence the total Segre class is given by $s(F) = e^{\Theta_C}$. In view of (2), the pullback of F along $f^\vee \colon X \to JC$ yields a vector bundle $\mathcal{E} \coloneqq (f^\vee)^*(F)$ on X which satisfies (4) in the lemma. This concludes the proof. □

Lemma 2.10. *Let (X,Θ) be a principally polarized abelian variety (over $\mathbb{C}$). Let $f \colon JC \to X$ be a morphism from a Jacobian of a curve and let $f^\vee : X \to JC$ be the dual with respect to the given principal polarizations. Assume that $f_*[C] = m \cdot [\Theta]^{g-1}/(g-1)! \in H_2(X,\mathbb{Z})$ for some positive integer m. Let Λ be a ring in which m is invertible. Then,*

$$(f_*^\vee H_1(X,\Lambda))^{\perp_{\Theta_C}} = \ker(f_* \colon H_1(JC,\Lambda) \to H_1(X,\Lambda)),$$

where $\perp_{\Theta_C}$ denotes the orthogonal complement with respect to Θ_C.

Proof. By Lemma 2.9, $\frac{1}{m} f_*^\vee$ splits the map $f_* \colon H_1(JC,\Lambda) \to H_1(X,\Lambda)$. Hence,

$$H_1(JC,\Lambda) = f_*^\vee H_1(X,\Lambda) \oplus \ker(f_* \colon H_1(JC,\Lambda) \to H_1(X,\Lambda)). \tag{2.4.1}$$

We will show that this decomposition is orthogonal with respect to Θ_C. To this end, let $\beta \in H_1(X,\mathbb{Z})$. The associated element of $H_1(X^\vee,\mathbb{Z}) = H_1(X,\mathbb{Z})^\vee$ is given by $\Theta(\beta,-)$, and its image via $f_*^\vee$ is given by

$$\Theta(\beta, f_*(-)) \in H_1(JC^\vee,\Lambda) = H_1(JC,\Lambda)^\vee.$$

It is clear that any class $\alpha \in \ker(f_*)$ lies in the kernel of the above linear form. This proves the inclusion

$$(f_*^\vee H_1(X,\Lambda))^{\perp_{\Theta_C}} \supset \ker(f_*\colon H_1(JC,\Lambda) \to H_1(X,\Lambda)).$$

Equality follows from (2.4.1) together with the fact that the restriction of Θ_C to the subspace $f_*^\vee H_1(X,\Lambda)$ is non-degenerate and unimodular (because m is invertible in Λ), see item (2) in Lemma 2.9. $\square$

Remark 2.11. In the notation of Lemma 2.10, if $Y \subset JC$ denotes the identity component of the kernel of $f\colon JC \to X$ endowed with its induced polarization, then the natural morphism $X \times Y \to JC$ is an isogeny of polarized abelian varieties whose degree is invertible in Λ. See [BGF23, Proposition 4.5] and [Voi24] for related results.

2.5. **Monodromy of abelian varieties and algebraicity of the minimal class in families.** We recall here some standard facts on the monodromy of families of abelian varieties and deduce some consequences on algebraicity of the minimal class in families that will be crucial for our approach; for further references and details, see for instance [EGFS25a, Section 2.3].

2.5.1. *Monodromy bilinear forms.* Let S be a finite set and let $\pi\colon X^\star \to (\Delta^\star)^S$ be a smooth projective family of principally polarized abelian varieties. We denote its fiber over $b \in (\Delta^\star)^S$ by (X_b, Θ). Fix a base point $t \in (\Delta^\star)^S$ and assume that for all $s \in S$, the monodromy operator T_s on $H_1(X_t,\mathbb{Z})$ about the s-th coordinate hyperplane of Δ^S is unipotent with nilpotent logarithm $N_s := T_s - \mathrm{id}$. We may then define a weight filtration

$$0 \subset W_{-2} \subset W_{-1} \subset W_0 = H_1(X_t,\mathbb{Z}),$$

where W_{-2} denotes the saturation of $\sum_{s\in S} \operatorname{im} N_s$ and $W_{-1} := \bigcap_{s\in S} \ker N_s$.

We will freely identify the theta divisor Θ on X_t with the associated bilinear form on $H_1(X_t,\mathbb{Z})$. This form induces an isomorphism

$$W_{-2}H_1(X_t,\mathbb{Z}) \xrightarrow{\sim} (\mathrm{gr}_0^W H_1(X_t,\mathbb{Z}))^*, \quad \alpha \mapsto \Theta(\alpha,-),$$

which allows us, as is well-known, to translate the nilpotent operator N_s into a bilinear form, as follows, see e.g. [EGFS25a, Definition 2.6]:

Definition 2.12. The s-th monodromy bilinear form B_s associated to the family $\pi\colon X^\star \to (\Delta^\star)^S$ is the bilinear form on $\mathrm{gr}_0^W H_1(X_t,\mathbb{Z})$, given by

$$x \otimes y \mapsto \Theta(N_s x, y).$$

Let us now assume in addition that π extends to a H-nodal morphism $X \to \Delta^S$, where $H \subset \Delta^S$ denotes the union of the coordinate hyperplanes. By a theorem of Clemens [Cle77, Theorem 7.36], the monodromy operator T_s from above is then automatically unipotent and so the aforementioned assumption is satisfied. Moreover, the above weight filtration is the weight filtration of the associated limit mixed Hodge structure. Let furthermore $\Gamma(X_0)$ be the dual complex of the central fiber X_0 of π. Then there is a canonical isomorphism

$$\mathrm{gr}_0^W H_1(X_t, \mathbb{Z}) \cong H_1(\Gamma(X_0), \mathbb{Z}).$$

The result is well-known for rational coefficients, but holds in the case of H_1 also for integral coefficients, see e.g. [EGFS25a, Proposition 5.10]. It follows that the monodromy bilinear forms B_s from above may be viewed, in a canonical manner, as bilinear forms on $H_1(\Gamma(X_0), \mathbb{Z})$, which we will denote with the same letter.

2.5.2. *Consequences of algebraicity of the minimal class in families.* Let $C \to \Delta^S$ be a semistable degeneration of curves, smooth over $(\Delta^\star)^S$. As above, we get for each $s \in S$ a monodromy bilinear form Q_s on

$$\mathrm{gr}_0^W H_1(C_t, \mathbb{Z}) \cong H_1(\Gamma(C_0), \mathbb{Z}).$$

From the hypothesis that C is smooth, each node of C_0 deforms to a node over exactly one coordinate hyperplane of Δ^S. Thus, the graph $G := \Gamma(C_0)$ is naturally S-colored, where we say that an edge has color s if the corresponding node deforms to a node of the general fiber of $C \to \Delta^S$ over the s-th coordinate hyperplane. Since $C \to \Delta^S$ is semistable, the Picard–Lefschetz formula shows that the above monodromy bilinear form Q_s is given by

$$Q_s = \sum_{e \in E_s} x_e^2,$$

where E_s denotes the set of edges of color s, and where x_e denotes the linear form on $H_1(\Gamma(C_0), \mathbb{Z})$ which is, up to a sign, uniquely determined by the edge e.

Assume in the set-up of Section 2.5.1 above, that we have a morphism $\iota \colon C \to X$ over Δ^S, such that

$$\iota_*[C_t] = m \cdot [\Theta]^{g-1}/(g-1)! \in H_2(X_t, \mathbb{Z}).$$

Note that ι induces a morphism of abelian schemes $f \colon JC|_{(\Delta^\star)^S} \to X|_{(\Delta^\star)^S}$.

Lemma 2.13. *Let ℓ be a prime with $\ell \nmid m$. There is a canonical direct sum decomposition*

$$\mathrm{gr}_0^W H_1(C_t) \cong \mathrm{gr}_0^W H_1(X_t) \oplus \ker(f_* \colon \mathrm{gr}_0^W H_1(C_t) \to \mathrm{gr}_0^W H_1(X_t)), \tag{2.5.1}$$

where homology is taken with $\mathbb{Z}_{(\ell)}$-coefficients. For each $s \in S$, the monodromy bilinear form Q_s on $\mathrm{gr}_0^W H_1(C_t)$ satisfies:

1. *the restriction of Q_s to $\mathrm{gr}_0^W H_1(X_t)$ agrees with m-times the monodromy bilinear form on $\mathrm{gr}_0^W H_1(X_t)$ induced by the family $X^\star \to (\Delta^\star)^S$;*
2. *the decomposition (2.5.1) is orthogonal with respect to Q_s.*

Proof. By Lemma 2.10, there is a direct sum decomposition

$$H_1(JC_t,\mathbb{Z}_{(\ell)})\cong H_1(X_t,\mathbb{Z}_{(\ell)})\oplus\ker(f_*\colon H_1(JC_t,\mathbb{Z}_{(\ell)})\to H_1(X_t,\mathbb{Z}_{(\ell)})) \tag{2.5.2}$$

which is orthogonal with respect to Θ_{C_t}. Here we used that $(f^\vee)^*\Theta_{C_t}=m\Theta$ (see Lemma 2.9), and m is invertible in $\mathbb{Z}_{(\ell)}$, so that $H_1(X_t,\mathbb{Z}_{(\ell)})\cong f_*^\vee H_1(X_t,\mathbb{Z}_{(\ell)})$.

The above decomposition respects the respective monodromy operators about the coordinate axis of Δ^S and hence in particular the respective weight filtrations. Taking gr_0^W, we thus get the decomposition (2.5.1). The fact that this decomposition is Q_s-orthogonal follows from the fact that the monodromy operator T_s, and hence the nilpotent operator N_s, respects the decomposition (2.5.2) together with the fact that the latter is orthogonal with respect to the principal polarization Θ_{C_t} (see Lemma 2.10). □

3. Admissible resolutions of S-colored morphisms

3.1. **S-colored D-quasi-nodal morphisms.** We aim to study base changes of nodal degenerations along morphisms $\tau\colon B'\to B$ such that the reduction of $\tau^{-1}(D)$ is snc. If S is the indexing set of the components of D, then this base change will turn out to satisfy the following auxiliary definitions, see Lemma 3.21 below.

Definition 3.1. Let (B,D) be a pair of a smooth variety B and an snc divisor D with components D_i, $i\in I$. A morphism $f\colon Y\to B$ is *D-quasi-nodal* if there is a set $\mathcal{C}$ of analytic charts that cover Y, such that for each $c\in\mathcal{C}$, $Y\to B$ is given by the normal form

$$\left\{\textstyle\prod_{i\in I}u_i^{a_{si}}=x_sy_s\mid s\in S'\right\}\subset U\times\Delta^{2|S'|}, \tag{3.1.1}$$

where u_i are regular functions on some analytic open $U\subset B$ with $D_i\cap U=\{u_i=0\}$, $a_{si}\geq 0$ are non-negative integers, and S' is a finite set.

The integers a_{si} may all vanish for a given $s\in S'$; this corresponds to a smooth factor in (3.1.1). Note also that U, a_{si}, and S' depend on the chart $c\in\mathcal{C}$; we will write U_c, $a_{si}(c)$, and S'_c if we want to emphasize the dependence on c.

Definition 3.2. Let S be a finite set. An *S-coloring* of a D-quasi-nodal morphism $f\colon Y\to B$ is the choice of a finite collection of effective Weil divisors $E_\alpha\subset Y$, $\alpha\in\Omega$, and an S-partition $\Omega=\bigsqcup_{s\in S}\Omega_s$, such that the following holds:

(1) The divisors E_α, $\alpha\in\Omega$, have pairwise distinct components, each E_α is contained in some $f^{-1}(D_i)$, and we have $Y\times_B D=\bigcup_{\alpha\in\Omega}E_\alpha$.
(2) For each $c\in\mathcal{C}$ as in (3.1.1), there is an inclusion $S'=S'_c\hookrightarrow S$ with the following property: For $\alpha\in\Omega_s$ with $E_\alpha\subset f^{-1}(D_i)$, we have that $E_\alpha\cap c$ is empty if $s\notin S'$ or $a_{si}=0$, and it is cut out by (u_i,x_s) or by (u_i,y_s) otherwise.

Remark 3.3. We ask item (2) for any atlas $\mathcal{C}$ as in Definition 3.1. However, it is easy to see that if the property holds for one atlas, then it holds for any atlas.

Remark 3.4. The local normal form (3.1.1) shows that f is flat. Condition (1) therefore determines for each $s \in S$ a canonical partition

$$\Omega_s = \bigsqcup_{i \in I} \Omega_{s,i}$$

by saying that $\alpha \in \Omega_s$ belongs to $\Omega_{s,i}$ if and only if $E_\alpha \subset f^{-1}(D_i)$. We will freely use this partition in what follows.

Remark 3.5. By item (2), the divisor $E_\alpha \cap c$ is either empty or cut out by (u_i, x_s), resp. (u_i, y_s). Note that this divisor is not necessarily irreducible. However, by (1), any of these ideals is the restriction of some divisor E_α. It follows that the components of $E_\alpha \cap c$ are regular and extend to global divisors. As a consequence, S-colored D-quasi-nodal morphisms are automatically strict D-quasi-nodal (cf. Definition 2.4).

The conditions in Definition 3.2 yield canonical S-colorings on the 1-skeleton $\Gamma^1(Y_p)$ of the fibers of f, as follows; this explains the phrase "S-coloring" in Definition 3.2.

Definition 3.6. In the notation of Definition 3.2, let $p \in B$. For $s \in S$, we say that an edge e of $\Gamma^1(Y_p)$ has *color* s if there are indices $\alpha_1, \alpha_2 \in \Omega_s$ such that e is given by the intersection of a component of $Y_p \cap E_{\alpha_1}$ with a component of $Y_p \cap E_{\alpha_2}$.

We define similarly an S-coloring on the graph $\Gamma^1(c_p)$ for any $c \in \mathcal{C}$. For each $p \in B$, the natural map $\Gamma^1(c_p) \to \Gamma^1(Y_p)$ is then a morphism of S-colored graphs.

Lemma 3.7. *Definition 3.6 yields a canonical S-coloring of $\Gamma^1(Y_p)$ for all $p \in B$. In a local chart c as in (3.1.1), the fiber c_p is given by the product of a smooth variety with*

$$\{x_s y_s = 0 \mid s \in S''\} \subset \Delta^{2|S''|},$$

where $S'' = \{s \in S' \mid \prod_i u_i^{a_{si}}(p) = 0\}$, and an edge e of $\Gamma^1(c_p)$ has color s if and only if $s \in S''$ and e corresponds to the intersection of a component of $x_s = 0$ with a component of $y_s = 0$.

Proof. This is clear from items (1) and (2) in Definition 3.2. $\square$

3.2. **Dual complexes of the fibers.** Let $Y \to B$ be S-colored D-quasi-nodal. Consider a chart c as in (3.1.1). For $p \in B$, we will denote by c_p the chart of Y_p induced by the chart $c \in \mathcal{C}$. This chart is empty if c does not meet Y_p. Otherwise, c_p is by the above assumption given by a product of a smooth variety with

$$\{0 = x_s y_s \mid \ s \in S''\} \subset \mathbb{A}^{2|S''|}, \tag{3.2.1}$$

where $S'' = \{s \in S' \mid \prod_{i \in I} u_i^{a_{si}}(p) = 0\}$, see Lemma 3.7.

Remark 3.8. Let $p \in B$. Items (1) and (2) in Definition 3.12 ensure that any component of c_p globalizes to a component of Y_p, cf. Remark 3.5. Taking suitable intersections, we see that, more generally, any intersection of components of c_p extends to a subscheme of Y_p. Since the components of c_p, as well as their intersections, are smooth, the same holds for the components of Y_p.

Lemma 3.9. *Let $Y \to B$ be an S-colored D-quasi-nodal degeneration. For each $c \in \mathcal{C}$, let $\Gamma(c_p)$ be the dual complex of the chart c_p given by (3.2.1). Then $\Gamma(c_p)$ is a cube of dimension $|S''|$ and the polyhedral cell structure of $\Gamma(Y_p)$ is induced by a quotient map*

$$\textstyle\bigsqcup_{c \in \mathcal{C}} \Gamma(c_p) \twoheadrightarrow \Gamma(Y_p), \tag{3.2.2}$$

which is injective on each cube $\Gamma(c_p)$ and identifies two cubes $\Gamma(c_p)$ and $\Gamma(c'_p)$ along d-dimensional faces F and F' via an isomorphism of polyhedral complexes $F \xrightarrow{\sim} F'$ if and only if F and F' correspond to the same component of the codimension d stratum $Y_p^{[d]}$ of Y_p.

Proof. Note first that any cell of the dual complex $\Gamma(Y_p)$ can be detected in some local chart (3.2.1). Hence, there is a natural surjection as in (3.2.2). The restriction of this map to each cube $\Gamma(c_p)$ is injective because of Remark 3.8. The description of the gluing follows from the definition of the dual complex. □

Remark 3.10. A priori the topological space $\Gamma(Y_p)$ may not be homeomorphic to a manifold. This behavior will not arise in our applications, where the charts (3.2.2) will define a cubical tiling of a real torus, cf. [EGFS25a, Figure 15].

3.3. Specialization maps.

Definition 3.11. Let $D \subset B$ be an snc divisor with components D_i, $i \in I$. Let $p \in D_J^\circ$ and $q \in D_{J'}^\circ$ be points of open strata of D. We say that p specializes to q if we are given a path $\gamma \colon [0,1] \to D_J$ with $\gamma(0) = p$ and $\gamma(1) = q$, such that $\gamma([0,1)) \subset D_J^\circ = D_J \setminus \bigcup_{i \in I \setminus J} D_i$ is contained in the open stratum $D_J^\circ \subset D_J$.

Let $f \colon Y \to B$ be a proper morphism whose fibers have double crossings in codimension 1 (i.e. the codimension one singularities are nodes) and such that f is topologically trivial over all open strata of D. Let $p \in D_J$ and $q \in D_{J'}$ be points such that p specializes to q via a continuous path γ, see Definition 3.11. Let $Y_\gamma \to [0,1]$ be the pullback of f via γ. Since $\gamma([0,1)) \subset D_J^\circ$, $Y_\gamma \to [0,1]$ restricts, topologically, over $[0,1)$ to the product of $Y_p = Y_{\gamma(0)}$ with $[0,1)$. Since f is proper, Y_γ retracts onto $Y_q = Y_{\gamma(1)}$. In this situation, we may define a continuous specialization map on graphs

$$\mathrm{sp} \colon \Gamma^1(Y_q) \longrightarrow \Gamma^1(Y_p)$$

as follows:

(1) if v is a component of Y_q, then $\mathrm{sp}(v)$ is the unique component of Y_p whose specialization along γ to Y_q contains the component v.
(2) if e is an edge of $\Gamma^1(Y_q)$, corresponding to a double locus where components v and v' meet, then consider the components $\mathrm{sp}(v)$ and $\mathrm{sp}(v')$ of Y_p. If $\mathrm{sp}(v) = \mathrm{sp}(v')$, then e is contracted to the vertex $\mathrm{sp}(v)$. Otherwise, $\mathrm{sp}(v)$ and $\mathrm{sp}(v')$ meet along a double locus and we let $\mathrm{sp}(e)$ be the corresponding edge of Y_q.

3.4. **Admissible modifications.** Recall from Section 2.1.4 that a refinement $\hat{G}$ of a graph G is a refinement of the corresponding cell complex. If G is S-colored, then $\hat{G}$ carries a canonical S-coloring.

Definition 3.12. Let B be a smooth variety with an snc divisor D with strata D_J, $J \subset I$. Let $Y \to B$ be an S-colored D-quasi-nodal morphism. We define an *admissible modification* $(g\colon Y' \to Y, \varphi)$ to be a pair, consisting of

(a) a projective modification $g\colon Y' \to Y$ for which $Y' \to B$ is D-quasi-nodal or D-semistable, and g induces an isomorphism over the complement of D, and
(b) for each $p \in B$, a refinement $\hat{\Gamma}^1(Y'_p)$ of the 1-skeleton $\Gamma^1(Y'_p)$, together with a morphism of S-colored graphs

$$\varphi_p\colon \hat{\Gamma}^1(Y'_p) \longrightarrow \Gamma^1(Y_p),$$

such that the following conditions hold:

(1) The dual complex $\Gamma(Y'_p)$ is given by a refinement of the cell structure of $\Gamma(Y_p)$ and the induced homeomorphism of topological spaces $\Gamma(Y_p) \xrightarrow{\approx} \Gamma(Y'_p)$ has the property that the following diagram is commutative up to homotopy:

$$\begin{tikzcd} \hat{\Gamma}^1(Y'_p) \arrow[r, "\varphi_p"] \arrow[d, hook] & \Gamma^1(Y_p) \arrow[d, hook] \\ \Gamma(Y'_p) & \Gamma(Y_p), \arrow[l, "\approx"'] \end{tikzcd}$$

where the vertical maps are the canonical inclusions.
(2) If $p \in D_J$ and $q \in D_{J'}$ are points of strata of D with $J \subset J'$, such that p specializes to q along a path γ (see Definition 3.11), then the following diagram of topological spaces commutes:

$$\begin{tikzcd} \hat{\Gamma}^1(Y'_p) \approx \Gamma^1(Y'_p) \arrow[d, "\varphi_p"] & \Gamma^1(Y'_q) \approx \hat{\Gamma}^1(Y'_q) \arrow[l, "\mathrm{sp}"'] \arrow[d, "\varphi_q"] \\ \Gamma^1(Y_p) & \Gamma^1(Y_q), \arrow[l, "\mathrm{sp}"'] \end{tikzcd}$$

where the horizontal maps are induced by the specialization maps from Section 3.3, and "$\approx$" indicates the canonical homeomorphism induced by the given refinements.

An admissible modification $(g\colon Y' \to Y, \varphi)$ such that Y' is D-semistable will also be called an *admissible resolution.*

Remark 3.13. Note that D-nodal and D-semistable morphisms are automatically flat and so is $Y' \to B$ in the above definition.

Remark 3.14. There is a natural map $\Gamma^1(Y_p) \hookrightarrow \hat{\Gamma}^1(Y'_p)$ of topological spaces, because the entire dual complex $\Gamma(Y'_p)$ is a refinement of $\Gamma(Y_p)$ and $\hat{\Gamma}^1(Y'_p)$ is a refinement of the 1-skeleton $\Gamma^1(Y'_p)$. In this paper, the collection $\varphi = (\varphi_p)_{p\in B}$ will always be constructed in such a way that φ_p is, up to homotopy, a retraction of this map, i.e. the composition $\Gamma^1(Y_p) \hookrightarrow \hat{\Gamma}^1(Y'_p) \xrightarrow{\varphi_p} \Gamma^1(Y_p)$ is homotopic to the identity, cf. item (1) in Definition 3.12.

The main result of this section is then the following theorem.

Theorem 3.15. *Let $D \subset B$ be an snc divisor with components D_i, $i \in I$. Let $Y \to B$ be an S-colored D-quasi-nodal morphism. Fix a total ordering on the set Ω from Definition 3.2. Then there is a canonical admissible modification $(g\colon Y' \to Y, \varphi)$ such that $Y' \to B$ is strict D-semistable; in particular, Y' is regular.*

Remark 3.16. Our resolution algorithm should be compared to the more general result in [ALT20, Theorem 2.7]. For our applications though, the collection $\varphi = (\varphi_p)_{p\in B}$ of maps of 1-skeleta is also crucial data.

The remainder of this section is devoted to a proof of the above theorem. We will first blow-up certain components E_α until the D-quasi-nodal morphism turns into a nearly D-nodal morphism, see Proposition 3.19 below. We will then choose certain common components of the Weil divisors E_α whose blow-ups in a carefully chosen order will produce an admissible resolution, see Proposition 3.20 below. Roughly speaking, the strategy of proof is to construct admissible resolutions in local charts and to exploit the global Weil divisors E_α from Definition 3.2 to show that our construction glues to give what we want globally.

Up to a slightly different ordering of the blow-ups, the same resolution (without the discussion of admissibility) is constructed in toric language in the survey [EGFS25a, §5.3]. (Strictly speaking, [EGFS25a, §5.3] only considers the case of a monomial base change of a strict $\bar{D}$-nodal morphism as in Lemma 3.21 below, which is however sufficient for the applications in this paper.)

3.4.1. *Reduction of the α-order.*

Definition 3.17. Let $Y \to B$ be an S-colored D-quasi-nodal morphism. For $\alpha \in \Omega_s$ and $p \in B$, we define the α-order $o_{\alpha,p}(c)$ of a chart c at $p \in B$ as follows. If $E_\alpha \cap c_p = \emptyset$, then $o_{\alpha,p}(c) = 0$. Otherwise, if c has normal form (3.1.1), then $o_{\alpha,p}(c)$ is the vanishing order of the function $\prod_j u_j^{a_{sj}}$ at p:

$$o_{\alpha,p}(c) \coloneqq \operatorname{ord}_p \left(\prod_j u_j^{a_{sj}} \right),$$

where we identified $S' = S'_c$ with a subset of S via item (2) in Definition 3.2. The α-order of Y is then defined as

$$o_\alpha(Y) \coloneqq \max_{c \in \mathcal{C},\, p \in B} o_{\alpha,p}(c),$$

where $\mathcal{C}$ is a set of charts that cover Y, as in Definition 3.1.

Note that if E_α contains p, then $o_{\alpha,p}(c)$ depends only on the color s of α.

Remark 3.18. In the notation of Definition 3.17, assume that E_α meets the chart c and $E_\alpha \in \Omega_{s,i}$, i.e. $E_\alpha \subset Y \times_B D_i$. Then E_α is cut out by (u_i, x_s) or (u_i, y_s), see item (2) in Definition 3.2. It follows that E_α is empty in a neighborhood of c_p if $o_{\alpha,p}(c) = 0$, and it is Cartier in such a neighborhood if $o_{\alpha,p}(c) = 1$.

Note that $Y \to B$ is nearly D-nodal (see Definition 2.4) if and only if $o_\alpha(Y) = 1$ for all $\alpha \in \Omega$ (where we assume $E_\alpha \neq \emptyset$ for all $\alpha \in \Omega$).

Proposition 3.19. *Let $f\colon Y \to B$ be an S-colored D-quasi-nodal morphism. Assume there is some $s \in S$ and $\alpha \in \Omega_s$ with $o_\alpha(Y) \geq 2$. Then the blow-up $g : Y' \to Y$ of the Weil divisor E_α is, for a canonically defined collection of maps $\varphi_p\colon \hat{\Gamma}^1(Y'_p) \to \Gamma^1(Y_p)$, $p \in B$, an admissible modification and $Y' \to B$ is S-colored D-quasi-nodal with indexing set $\Omega' \coloneqq \Omega \sqcup \{\beta\}$ for some $\beta \in \Omega'$ such that*

$$o_\alpha(Y') = o_\beta(Y') = o_\alpha(Y) - 1$$

and $o_\gamma(Y') = o_\gamma(Y)$ for $\gamma \in \Omega \setminus \{\alpha\}$.

Proof. Since $\alpha \in \Omega_s$, item (2) in Definition 3.2 implies that $E_\alpha \subset Y \times_B D_i$ for some $i \in I$. In the notation of Remark 3.4, this is saying that $\alpha \in \Omega_{s,i}$.

We proceed in several steps.

Step 1. Blow-up of local charts.

Let c be a chart with normal form

$$\left\{ \prod_{j \in I} u_j^{a_{tj}} = x_t y_t \;\middle|\; t \in S' \right\} \subset U \times \Delta^{2|S'|}.$$

Assume that locally on c, E_α is non-empty. By item (2) in Definition 3.2, we may regard $S' \subset S$ as a subset. Moreover, since $\alpha \in \Omega_{s,i}$ and E_α is non-empty in the chart c, it is cut out by (u_i, x_s) or by (u_i, y_s). Up to replacing x_s by y_s, we can without loss of generality assume that it is cut out by (u_i, x_s). The blow-up $c' \to c$ of the ideal (u_i, x_s) is covered by two charts, denoted by (c, x_s) and (c, u_i), respectively, as follows. The (c, x_s)-chart is explicitly given by

$$\left\{u_i = x_s u_i', \quad \textstyle\prod_{j\in I\setminus\{i\}} u_j^{a_{tj}} = x_t y_t \mid t \in S'\right\}, \tag{3.4.1}$$

while the (c, u_i)-chart is given by

$$\left\{u_i^{a_{si}-1} \textstyle\prod_{j\in I\setminus\{i\}} u_j^{a_{sj}} = x_s' y_s, \quad \prod_{j\neq i} u_j^{a_{tj}} = x_t y_t \mid t \in S'\right\}. \tag{3.4.2}$$

Altogether this shows that $Y' \to B$ is a D-quasi-nodal morphism.

Step 2. $Y' \to B$ is canonically an S-colored D-quasi-nodal morphism.

We define $\Omega_s' := \Omega_s \sqcup \{\beta\}$ for a new element β and $\Omega_t' := \Omega_t$ for $t \in S \setminus \{s\}$. The indexing set of Y' needed in Definition 3.2 is then defined as $\Omega' := \Omega \sqcup \{\beta\}$, together with the natural S-partition: $\Omega' = \bigsqcup_{t\in S} \Omega_t'$. For $\gamma \in \Omega'$ with $\gamma \neq \beta$, we define E_γ' to be the proper transform of E_γ in Y'. Moreover, we define E_β' to be the reduced divisor contracted by $g : Y' \to Y$. This divisor is contained in $Y' \times_B D_i$ (since $E_\alpha \subset f^{-1}(D_i)$), and so $E_\beta' \in \Omega_{s,i}'$. We have thus defined a collection of Weil divisors on Y', indexed by the sets Ω_t', $t \in S$, which clearly satisfy item (1) in Definition 3.2.

In the (c, x_s)-chart (3.4.1), we have:

$$E_\alpha' = \emptyset \qquad \text{and} \qquad E_\beta' = \{u_i = x_s = 0\} \tag{3.4.3}$$

and in the (c, u_i)-chart (3.4.2), we have

$$E_\alpha' = \{u_i = x_s' = 0\} \qquad \text{and} \qquad E_\beta' = \{u_i = y_s = 0\}. \tag{3.4.4}$$

(This uses the assumption $o_{\alpha,p}(c) \geq 2$.) Moreover, for any $\gamma \in \Omega_t'$ with $t \in S \setminus \{s\}$, we know by assumptions that E_γ is either trivial on c or it agrees with the vanishing locus of some ideal of the form (u_j, x_t), resp. (u_j, y_t), for some $j \in I$. The same local description holds for the proper transform E_γ' in the charts (3.4.1) and (3.4.2).

In order to prove item (2) in Definition 3.2 for Y', it thus remains to show that the Weil divisor cut out by (u_i, u_i') in the chart (3.4.1) is the restriction of $E_{\alpha'}'$ for some $\alpha' \in \Omega_s'$. The natural candidate for this is the index $\alpha' \in \Omega_s$, such that $E_{\alpha'}$ is cut out by (u_i, y_s) in the chart (3.4.1); which exists because $Y \to B$ is S-colored, see items (1) and (2) in Definition 3.2. In (3.4.1), we have

$$y_s = x_s^{a_{si}-1} (u_i')^{a_{si}} \prod_{j\in I\setminus\{i\}} u_j^{a_{sj}},$$

while $x_s = 0$ cuts out the union of the exceptional divisor with the proper transform E'_α of E_α. Since $u_i = x_s u'_i$, the proper transform of the vanishing locus of (u_i, y_s) is in the chart (3.4.1) cut out by $(u'_i, u_i^{a_{si}-1} u'_i \prod_{j \in I \setminus \{i\}} u_j^{a_{sj}}) = (u'_i)$. This ideal agrees with (u_i, u'_i), as we want. Altogether, we have thus shown that item (2) in Definition 3.2 holds and so $Y' \to B$ is an S-colored D-quasi-nodal morphism in a canonical way, as claimed.

Step 3. Cell structure of $\Gamma(Y'_p)$.

Let $p \in B$. If $p \notin D_i$, then $Y'_p \to Y_p$ is an isomorphism and hence $\Gamma(Y'_p) = \Gamma(Y_p)$ as cell complexes. Let now $p \in D_i$. We have a presentation

$$\bigsqcup_c \Gamma(c_p) \twoheadrightarrow \Gamma(Y_p)$$

of $\Gamma(Y_p)$ by the cuboids $\Gamma(c_p)$, where c runs through all charts that meet Y_p, see Lemma 3.9. If $c \in \mathcal{C}$ is such a chart, then we consider the blow-up $c' \to c$ induced by $Y' \to Y$. Since $c' \to c$ is an isomorphism in a neighborhood of c_p if $o_{\alpha,p}(c) \leq 1$, we can further assume that $o_{\alpha,p}(c) \geq 2$.

The charts (3.4.1) and (3.4.2) cover c'. We thus obtain a surjection

$$\Gamma((c, x_s)_p) \sqcup \Gamma((c, u_i)_p) \twoheadrightarrow \Gamma(c'_p) \tag{3.4.5}$$

of cell complexes. Composing the disjoint union of the maps (3.4.5) for $c \in \mathcal{C}$ with the natural surjection $\sqcup_c \Gamma(c'_p) \twoheadrightarrow \Gamma(Y'_p)$ yields the cell structure of $\Gamma(Y'_p)$.

The map (3.4.5) identifies $\Gamma(c'_p)$ with the cell complex given by gluing the cube $\Gamma((c, x_s)_p)$ to $\Gamma((c, u_i)_p)$ along their common codimension 1 face F, and can be described as follows. The chart c_p is isomorphic to a product of a smooth variety with

$$\{x_t y_t = 0 \mid t \in S''\} \subset \Delta^{2|S''|}$$

for some subset $S'' \subset S'$. (Here $s \in S''$ because $o_{\alpha,p}(c) \geq 2$ and $\alpha \in E_s$.) By Remark 3.8, each component of c_p extends to a divisor on Y. The proper transform of this divisor yields a divisor in Y' which in turn yields a component of c'_p. This defines an inclusion $\Gamma^0(c_p) \hookrightarrow \Gamma^0(c'_p)$, which in turn extends uniquely to a linear homeomorphism of cuboids $\Gamma(c_p) \approx \Gamma(c'_p)$. The vertices of $\Gamma^0(c'_p)$ that are not contained in $\Gamma^0(c_p)$ are precisely those of the faces of $\Gamma((c, x_s)_p)$ and $\Gamma((c, u_i)_p)$ that are glued in (3.4.5). They can intrinsically be defined as those components of c'_p that are contracted via $c'_p \to c_p$.

In terms of the coloring of the 1-skeleton of $\Gamma(c_p)$, the refinement in (3.4.5) corresponds to adding a vertex at the middle of each edge of color s and to subdivide the cuboid $\Gamma(c_p) \approx \Gamma(c'_p)$ along the wall that is generated by those vertices. See Figure 1.

Step 4. $g : Y' \to Y$ is admissible for a canonically defined collection $\varphi = \{\varphi_p\}_{p \in B}$ of maps $\varphi_p \colon \Gamma^1(Y'_p) \to \Gamma^1(Y_p)$.

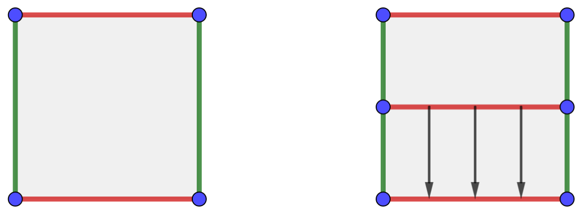

FIGURE 1. Left: Cube $\Gamma(c_p)$ of the dual complex $\Gamma(Y_p)$. Right: Cuboid refinement $\Gamma(c'_p)$ of this cube, corresponding to the fiber of the blowup $Y'_p \to Y_p$. Map $\varphi_p\colon \Gamma(c'_p) \to \Gamma(c_p)$ is depicted by arrows.

We have already seen that there is a canonical way in which we can, for any $p \in B$, identify $\Gamma(Y'_p)$ with a refinement of the cell structure of $\Gamma(Y_p)$. This gives a homeomorphism $\Gamma(Y_p) \xrightarrow{\approx} \Gamma(Y'_p)$, which restricts to a continuous map on 1-skeleta $\Gamma^1(Y_p) \to \Gamma^1(Y'_p)$.

We define $\hat{\Gamma}^1(Y'_p) := \Gamma^1(Y'_p)$ (no further refinement of the graph is necessary in this case) with the S-coloring induced from the one of $\Gamma^1(Y'_p)$ from Step 2 and Definition 3.6. We then aim to construct a map of S-colored graphs $\varphi_p\colon \Gamma^1(Y'_p) \to \Gamma^1(Y_p)$ which is homotopic to a retraction of the continuous map $\Gamma^1(Y_p) \to \Gamma^1(Y'_p) \subset \Gamma(Y'_p)$.

We start with the local problem in the chart c from Step 1 and we assume that $o_{\alpha,p}(c) \geq 2$. We then aim to define a canonical map

$$\varphi_p\colon \Gamma^1(c'_p) \longrightarrow \Gamma^1(c_p)$$

of colored graphs with the desired properties. In the present case, it turns out that one can define φ_p easily on the entire dual complexes to obtain maps $\Gamma(c'_p) \to \Gamma(c_p)$. A priori, there are two obvious choices: Via the refinement (3.4.5), we could contract either $\Gamma((c, x_s)_p)$ or $\Gamma((c, u_i)_p)$, by contracting its s-colored edges, and identifying the result canonically with $\Gamma^1(c_p)$. The fact that the divisor E_α that is blown-up is part of the data in the proposition, allows us to pick one of the choices in a canonical way: the proper transform $E'_\alpha \subset Y'$ is disjoint from the chart (c, x_s), but meets the chart (c, u_i), see (3.4.3). This yields a canonical map of cell complexes

$$\psi_p\colon \Gamma(c'_p) \longrightarrow \Gamma(c_p),$$

by collapsing in the presentation (3.4.5) edges of color s of $\Gamma((c, u_i)_p)$. See Figure 1. The above map of cell complexes induces a canonical map on 1-skeleta: $\varphi_p\colon \Gamma^1(c'_p) \to \Gamma^1(c_p)$. Our construction is canonical and in particular compatible with localization. Hence, it glues to give a unique map of S-colored graphs

$$\varphi_p\colon \Gamma^1(Y'_p) \longrightarrow \Gamma^1(Y_p),$$

that we denote by the same letter.

Each cuboid $\Gamma(c_p)$ is embedded into $\Gamma(Y_p)$ (see Lemma 3.9), and φ_p maps $\Gamma^1(c'_p)$ to $\Gamma^1(c_p)$. Moreover, there exists a canonical homotopy between the continuous self-map of the cuboid $\Gamma(c_p)$ induced by ψ_p, and the identity on $\Gamma(c_p)$. These homotopies glue to define the homotopy required in the diagram in item (1) of Definition 3.12.

To check the compatibility under specialization stated in item (2) of Definition 3.12, it suffices to prove that this compatibility holds in each chart. Let $p \in D_J$ and $q \in D_{J'}$ be points as in item (2) of Definition 3.12, so that p specializes to q. Let further c be a chart of Y. If $o_{\alpha,q}(u) \leq 1$, then also $o_{\alpha,p}(u) \leq 1$ and $\varphi_p\colon \Gamma^1(c'_p) \to \Gamma^1(c_p)$ and $\varphi_q : \Gamma^1(c'_q) \to \Gamma^1(c_q)$ are isomorphisms, hence the commutativity in item (2) is clear. If $o_{\alpha,p}(c) \geq 2$, then also $o_{\alpha,q}(c) \geq 2$ and the specialization maps in the diagram

$$\begin{tikzcd}
\Gamma^1(c'_p) \arrow[d, "\varphi_p"] & \Gamma^1(c'_q) \arrow[l, "\mathrm{sp}"'] \arrow[d, "\varphi_q"] \\
\Gamma^1(c_p) & \Gamma^1(c_q), \arrow[l, "\mathrm{sp}"']
\end{tikzcd}$$

are isomorphisms and the commutativity is clear. Finally, if $o_{\alpha,q}(c) \geq 2$, but $o_{\alpha,p}(c) = 1$, then the left most arrow as well as the lower horizontal arrow in the above diagram are isomorphisms, while the specialization map $\Gamma^1(c'_q) \to \Gamma^1(c'_p)$ identifies to the map φ_q via these identifications. Here we use the fact that ψ_p contracts edges of color s of $\Gamma((c, u_i)_p)$, hence it contracts edges that correspond to intersections of the proper transform of E_α with the exceptional divisor of $g : Y' \to Y$. Altogether, we have thus proven the compatibility stated in item (2) of Definition 3.12. This concludes the proof of the Proposition 3.19. $\square$

3.4.2. *Small resolution of S-colored nearly D-nodal degenerations.*

Proposition 3.20. *Let $Y \to B$ be an S-colored D-quasi-nodal morphism. Assume $o_\alpha(Y) \leq 1$ for all $\alpha \in \Omega$, i.e. $Y \to B$ is nearly D-nodal. Then a total order on Ω induces in a canonical manner an admissible modification $(g\colon Y' \to Y, \varphi)$, such that $Y' \to B$ is strict D-semistable.*

Proof. We proceed in several steps.

Step 1. Local resolution in a chart c.

Let $c \in \mathcal{C}$ be a chart with normal form (3.1.1). By assumption, $o_\alpha(Y) \leq 1$ for all $\alpha \in \Omega$. Hence, $\sum_{i \in I} a_{si}$ is either 0 or 1 and c is (a smooth base change of):

$$\{u_i = x_s y_s \mid i \in I',\ s \in S_i\} \subset U \times \Delta^{2|S''|}, \tag{3.4.6}$$

where $I' \subset I$ is the subset of $i \in I$ with $a_{si} = 1$ for some s, $S_i = \{s \in S \mid a_{si} = 1\}$ and $S'' \subset S'$ is the subset of $s \in S'$ with $a_{si} = 1$. Note that the subsets S_i are pairwise disjoint, hence $S'' = \cup_{i\in I'} S_i$ is a partition. Let $p \in U$. Up to shrinking U, we can assume that u_i with $i \in I'$ can be extended to analytic coordinates on U and that $u_i(p) = 0$ for all $i \in I'$. Then the above normal form is locally analytically isomorphic to the product of a polydisc with the product

$$\prod_{i\in I'} \{u_i = x_s y_s \mid s \in S_i\} \subset \prod_{i\in I'} \Delta_{u_i} \times \Delta^{2|S_i|}.$$

In other words, $Y \to B$ is nearly D-nodal, see Definition 2.4.

The above product is resolved by resolving each factor independently. For $i \in I$, we denote by c_i the chart

$$\{u_i = x_s y_s \mid s \in S_i\} \tag{3.4.7}$$

of the i-th factor. Note that this is a family over a curve whose central fiber is a union of $2^{|S_i|}$-many components; the dual complex $\Gamma(c_{i,p})$ of the central fiber is isomorphic to the cube $[0,1]^{|S_i|}$ of dimension $|S_i|$, given as the product of edges of color $s \in S_i$ which correspond to the node $x_s = y_s = 0$. The singularities of c_i admit a small resolution $c_i' \to c_i$, given by blowing-up the non-Cartier components of the central fiber of c_i repeatedly (unique up to a choice of an order of the components to be blown-up). A local computation shows that the result is semi-stable over Δ_{u_i}, i.e. it will be given by local equations of the form $u_i = z_1 z_2 \cdots z_m$ for some m, cf. [EGFS25a, §5.3]. Moreover, the small resolution $c_i' \to c_i$ induces a bijection on the set of components of each fiber. See Figure 2.

Step 2. For each $p \in B$, there is a canonical partial ordering on the vertices $\Gamma^0(Y_p)$ which for each chart $c \in \mathcal{C}$ is a total order on $\Gamma^0(c_p)$.

Recall from Remark 3.4 the partition $\Omega_s = \sqcup_{i\in I} \Omega_{s,i}$. We then define $\Omega_i := \sqcup_{s\in S} \Omega_{s,i}$ and get a second partition $\Omega = \sqcup_{i\in I} \Omega_i$ of Ω. Without loss of generality, D is nonempty and so are its components: $D_i \neq \emptyset$ for all $i \in I$. Hence, item (1) in Definition 3.2 implies that $\Omega_i \neq \emptyset$. The total order on $\Omega = \sqcup_{i\in I} \Omega_i$ thus induces a total ordering on I by saying that $i < j$ if there is an element in Ω_i that is smaller than all elements in Ω_j. Up to replacing S by a subset, we can without loss of generality assume that $\Omega_s \neq \emptyset$ for all $s \in S$. For the same reason as above, the total order on $\Omega = \sqcup_{s\in S} \Omega_s$ then induces a total ordering on S.

Consider the chart c in (3.4.6). Then $u_i = 0$ cuts out the components of $c \times_B D_i$. Each of these components can, by items (1) and (2) in Definition 3.2, be described as intersections of a collection of divisors

$$\bigcap_{s\in S_i} E_{\alpha_s} \qquad \text{for some} \qquad (\alpha_s)_{s\in S_i} \in \prod_{s\in S_i} \Omega_{s,i}.$$

The total ordering on S induces a total ordering on its power set, with the property that $A < B$ whenever $A, B \subset S$ are subsets with $|A| < |B|$; subsets of the same cardinality are ordered via the lexicographic ordering. Moreover, the total ordering on Ω induces a total ordering on each of its subsets and on powers of these sets via the lexicographic order. Using this, we get for each $i \in I$ a total ordering on the following set:

$$\begin{aligned}\mathfrak{S}_i &:= \coprod_{A \in \mathcal{P}(S)} \prod_{s \in A} \Omega_{s,i} \\ &= \{(\alpha_s)_{s\in A} \mid A \in \mathcal{P}(S), \quad \alpha_s \in \Omega_{s,i} \text{ for all } s \in A\}.\end{aligned}$$

Note that this ordering has the property that $(\alpha_s)_{s\in A} < (\beta_s)_{s\in A'}$ whenever $|A| < |A'|$.

For $(\alpha_s)_{s\in A} \in \mathfrak{S}_i$, we define the divisor

$$E_{(\alpha_s)_{s\in A}} \quad \text{as the common components of } E_{\alpha_s} \text{ with } s \in A. \tag{3.4.8}$$

In the chart c as in (3.4.6), the components of $c \times_B D_i$ are then uniquely identified with an element $(\alpha_s)_{s\in S_i} \in \mathfrak{S}_i$.

The total ordering on $\mathfrak{S}_i$ thus induces for all $p \in B$ a total ordering on the vertices $\Gamma^0(c_{i,p})$, where c_i is the chart in (3.4.7). The total ordering of I, together with the lexicographic order on tuples, thus defines a total ordering on $\Gamma^0(c_p) = \prod_{i\in I'} \Gamma^0(c_{i,p})$, because $c = \prod_{i\in I'} c_i$. These total orderings in charts depend only on the total ordering of Ω and I; they therefore glue to give a well-defined partial ordering on $\Gamma^0(Y_p)$ for each $p \in B$.

Step 3. Global resolution $g : Y' \to Y$.

We construct g by repeatedly blowing-up divisors of the form (3.4.8). We determine the order as follows. We use the total ordering of I induced by the one of Ω (see Step 2) and go from the largest to the smallest element. For given $i \in I$, we consider the totally ordered set $\mathfrak{S}_i$ from Step 2 and go again from the largest to the smallest element. For each $(\alpha_s)_{s\in A} \in \mathfrak{S}_i$ in the given ordering, we then blow-up the divisor $E_{(\alpha_s)_{s\in A}}$.

In the local chart (3.4.6), the blow-up of $E_{(\alpha_s)_{s\in A}}$ only affects the i-th factor of $c = \prod_{j\in I'} c_j$; moreover, c_i as in (3.4.7) is affected as follows: If $|A| > |S_i|$ or $|A| = |S_i|$ but $A \neq S_i$, then $E_{(\alpha_s)_{s\in A}}$ is empty in c and the chart is not affected. If $A = S_i$ and $(\alpha_s)_{s\in S_i}$ runs through all respective tuples, then the chart c_i will be resolved, as explained in Step 1. Afterwards, c_i' is regular and further blow-ups of Weil divisors as in (3.4.8) with $(\alpha_s)_{s\in A} \in \mathfrak{S}_i$ will not alter the chart anymore.

Altogether we have thus seen that g is a small resolution which restricts to the resolution described in Step 1 in charts. In particular, for all $p \in B$, $Y'_p \to Y_p$ induces a bijection on irreducible components. Moreover, the local analysis in charts shows that $Y' \to B$ is D-semistable and that the natural map $Y' \times_B D_i \to Y \times_B D_i$ also induces a bijection on the set of irreducible components. Since $Y \to B$ is S-colored D-quasi-nodal, it follows that the components of $Y \times_B D_i$ are smooth (i.e. there are no self-intersections) and so the local analysis shows that the same holds for the components of $Y' \times_B D_i$. It follows that $Y' \to B$ is strict D-semistable.

Step 4. First properties needed in Definition 3.12.

Note first that $Y' \to B$ is (strict) D-semistable by Step 3 and $Y' \to Y$ is an isomorphism over $B \setminus D$ because we have blown-up components of E_α and these divisors are supported on $Y \times_B D$ by (1) in Definition 3.2.

Next, we aim to describe $\Gamma^1(Y'_p)$ as a refinement of $\Gamma^1(Y_p)$. For $p \in B$, consider the presentation

$$\bigsqcup_{c \in \mathcal{C}} \Gamma(c_p) \twoheadrightarrow \Gamma(Y_p)$$

of the cell structure of $\Gamma(Y_p)$, given by (3.2.2). Then the dual complex $\Gamma(Y'_p)$ admits a presentation

$$\bigsqcup_{c \in \mathcal{C}} \Gamma(c'_p) \twoheadrightarrow \Gamma(Y'_p), \tag{3.4.9}$$

where c'_p is the fiber at p of the small resolution c' of the chart c, induced by $g : Y' \to Y$. We have seen in Step 1 that $\Gamma(c'_p) = \prod_{i \in I'} \Gamma(c'_{i,p})$ is a refinement of the cube $\Gamma(c_p) = \prod_{i \in I'} \Gamma(c_{i,p})$, given by the fact that each factor $\Gamma(c'_{i,p})$ is a simplicial refinement of the cuboid $\Gamma(c_{i,p})$, given by tiling the cuboid $\Gamma(c_{i,p})$ into standard simplicies (determined by the order of the blow-up). These refinements of the cubes $\Gamma(c_p)$ are compatible for different charts c, because the blow-ups performed are defined globally. This identifies the cell structure of $\Gamma(Y'_p)$ with a refinement of the cell structure on $\Gamma(Y_p)$. We further note that this refinement is an isomorphism on the 0-skeleta, because $Y'_p \to Y_p$ induces a bijection on the sets of irreducible components, see Step 1.

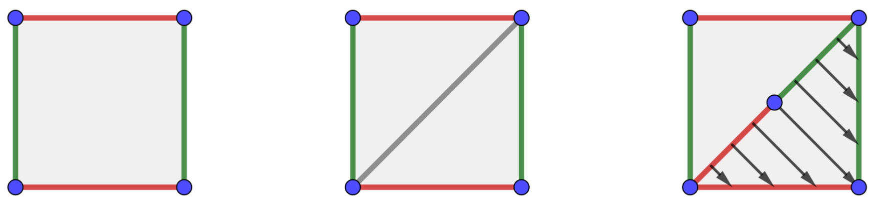

FIGURE 2. Left: Cube $\Gamma(c_p)$ of $\Gamma(Y_p)$. Middle: Dual complex $\Gamma(c'_p)$ in the fiber $Y'_p \to Y_p$ of the resolution with colorless (grey) diagonal edge. Right: Subdivision $\hat{\Gamma}^1(c'_p)$ of $\Gamma^1(c'_p)$ and its coloring. Map $\varphi_p \colon \hat{\Gamma}^1(c'_p) \to \Gamma^1(c_p)$ of colored graphs is depicted by arrows.

Step 5. Definition of $\hat{\Gamma}^1(Y'_p)$ and φ_p.

By (3.4.9), the 1-skeleton $\Gamma^1(Y'_p)$ is covered by the 1-skeleta $\Gamma^1(c'_p)$ with $c \in \mathcal{C}$. Since $Y \to B$ is S-colored D-quasi-nodal, $\Gamma^1(c_p)$ is an S-colored graph. Moreover, two edges of $\Gamma^1(c_p)$ have the same color if and only if they are parallel edges of the cuboid $\Gamma(c_p)$. In the above notation, $\Gamma(c_p) = \prod_{i \in I'} \Gamma(c_{i,p})$ and $\Gamma(c'_p) = \prod_{i \in I'} \Gamma(c'_{i,p})$. Here, $\Gamma(c'_{i,p})$ is a refinement of the cuboid $\Gamma(c_{i,p})$ given by standard simplices spanned by certain subsets of vertices of $\Gamma(c_{i,p})$. The 1-skeleton $\Gamma^1(c'_{i,p})$ has the same vertices as $\Gamma^1(c_{i,p})$ but contains additional edges, which we refer to as *diagonal* edges. Any two such diagonal edges will meet only at vertices of $\Gamma^1(c_{i,p})$. See Figure 2.

By the above description, $\Gamma^1(c'_p)$ identifies to the 1-skeleton of the product $\prod_{i \in I'} \Gamma^1(c'_{i,p})$. Hence, any edge e of $\Gamma^1(c'_p)$ is given by a product of an edge e_{i_0} of $\Gamma^1(c'_{i_0,p})$ for some $i_0 \in I'$ with vertices of $\Gamma^1(c'_{i,p})$ with $i \neq i_0$. We will say that e is a diagonal edge, if e_{i_0} is a diagonal edge of $\Gamma^1(c'_{i_0,p})$. Note moreover that any edge which is not diagonal is an edge that is contained in $\Gamma^1(c_p) \subset \Gamma^1(c'_p)$ (where we identify $\Gamma(c'_p)$ canonically with a refinement of $\Gamma(c_p)$) and hence it carries a unique color $s \in S$. Hence, $\Gamma^1(c'_p)$ is naturally a partially S-colored graph by declaring that diagonal edges are colorless. These partial S-colorings are compatible for different charts (because the S-coloring of $\Gamma^1(c_p)$ comes from a global S-coloring of $\Gamma^1(Y_p)$) and hence induce the structure of a partially S-colored graph on $\Gamma^1(Y'_p)$.

Note that $\Gamma^1(Y'_p)$ and $\Gamma^1(Y_p)$ have the same vertices and they agree on colored edges. Let now e be a colorless edge of $\Gamma^1(Y'_p)$. By (3.4.9), there is a chart c of Y such that e is contained in $\Gamma^1(c'_p)$. By Step 2, we have a partial ordering on the edges $\Gamma^0(Y'_p)$, which induces a total order on $\Gamma^0(c'_p) = \Gamma^0(c_p)$. Using this, we get a canonical orientation of e, directing from the smaller to the larger edge. We then consider 1-chains $\gamma \in C_1(\Gamma^1(c_p), \mathbb{Z})$ with $\partial e = \partial \gamma$. The length of γ is the minimal number of edges, needed to write γ as a linear combination of oriented edges. We then define the taxicab-norm $l(e)$ of e as the smallest number such that a 1-chain γ as above of length $l(e)$ exists. The choice of such a 1-chain γ may not be unique, but there will be a unique one, that we call a *geodesic path*, with the property that the vertices that are covered by the path γ are minimal in the lexicographic order induced by the total order on $\Gamma^0(c_p)$ from Step 2. Compatibility of the tiling $\Gamma(c'_p)$ of $\Gamma(c_p)$ and ordering of its vertices, for different charts c, shows that the above definition of $l(e)$ and the geodesic path are independent of the chart.

We may then divide each diagonal edge e of $\Gamma^1(Y'_p)$ (which comes with a natural orientation) into a chain $\hat{e}$ of $l(e)$ consecutive oriented edges such that $\hat{e}$, viewed as a 1-chain, is isomorphic to the unique geodesic path γ with $\partial e = \partial \gamma$. Since each edge of γ is colored, this induces a unique coloring of the edges of $\hat{e}$. Replacing each diagonal edge e of $\Gamma^1(Y'_p)$ by $\hat{e}$ then defines a refinement $\hat{\Gamma}^1(Y'_p)$ of $\Gamma^1(Y'_p)$ together with a canonical

S-coloring. Moreover, there is a natural morphism of S-colored graphs

$$\varphi_p\colon \hat{\Gamma}^1(Y'_p) \longrightarrow \Gamma^1(Y_p),$$

which is the identity on the subgraph $\Gamma^1(Y'_p) \subset \hat{\Gamma}^1(Y'_p)$ induced by the refinement (3.4.9), and which maps for each diagonal edge e of $\Gamma^1(Y'_p)$ the chain of edges $\hat{e}$ to the unique geodesic path γ with $\partial e = \partial\gamma$. See Figure 2. This concludes Step 5.

Step 6. φ_p respects the "product-structure".

For further reference below, we note that if $c = \prod_{i\in I'} c_i$, and e is a diagonal edge of $\Gamma^1(c'_p)$, then e is the product of an edge of $\Gamma^1(c_j)$ for some $j \in I'$ with vertices in the remaining factors. The given product of vertices in the factors $\Gamma^1(c_i)$ with $i \neq j$ define a face $\Gamma^1(c_j) \hookrightarrow \prod_{i\in I'} \Gamma^1(c_i)$ and any shortest length path γ with $\partial\gamma = \partial e$ will be contained in that face. We will refer to this property as to saying that

$$\varphi_p\colon \hat{\Gamma}^1(c'_p) \longrightarrow \Gamma^1(c_p) \tag{3.4.10}$$

respects the natural "product-structure" induced by $\Gamma(c'_p) = \prod_{i\in I'} \Gamma(c'_{p,i})$ and $\Gamma(c_p) = \prod_{i\in I'} \Gamma(c_{p,i})$.

Step 7. Items (1) and (2) in Definition 3.12.

Recall that the construction of φ_p is local in the charts of Y. In particular, φ_p maps $\hat{\Gamma}^1(c'_p)$ to $\Gamma^1(c_p)$ and, in this case, this map is a retract of the natural inclusion $\Gamma^1(c_p) \hookrightarrow \hat{\Gamma}^1(c'_p)$. Hence, $\varphi_p\colon \hat{\Gamma}^1(Y'_p) \to \Gamma^1(Y_p)$ is a retract and so the diagram in item (1) of Definition 3.12 commutes in the present case on the nose (not only up to homotopy).

To check item (2), let $p \in D_J$ and $q \in D_{J'}$ be points of strata with $J \subset J'$, such that p specializes to q. Let $c \in \mathcal{C}$ and recall from Step 1 that $\Gamma(c_q) = \prod_{i\in I'} \Gamma(c_{i,q})$, where c_i is as in (3.4.7). Moreover, $\Gamma(c_p) = \prod_{i\in I''} \Gamma(c_{i,p})$, where $I'' \subset I'$ is the subset of indices i with $u_i(p) = 0$ (equivalently, with $p \in D_i$). The specialization maps

$$\mathrm{sp}\colon \Gamma^1(c_q) \longrightarrow \Gamma^1(c_p) \qquad \text{and} \qquad \mathrm{sp}\colon \hat{\Gamma}^1(c'_q) \longrightarrow \hat{\Gamma}^1(c'_p)$$

are induced by the natural projection maps

$$\mathrm{pr}_{I''}\colon \prod_{i\in I'} \Gamma^1(c_{i,q}) \longrightarrow \prod_{i\in I''} \Gamma^1(c_{i,q}) \cong \prod_{i\in I''} \Gamma^1(c_{i,p}) \quad \text{and}$$
$$\mathrm{pr}_{I''}\colon \prod_{i\in I'} \hat{\Gamma}^1(c'_{i,q}) \longrightarrow \prod_{i\in I''} \hat{\Gamma}^1(c'_{i,q}) \cong \prod_{i\in I''} \hat{\Gamma}^1(c'_{i,p}).$$

Recall that an edge of $\hat{\Gamma}^1(c'_p)$ is given by the product of an edge e_j of some $\hat{\Gamma}(c'_{j,p})$, $j \in I'$, with a product of vertices v_i in the remaining factors. By Step 6, φ_p restricts to a map of the form

$$\varphi_p\colon \hat{\Gamma}^1(c'_{j,p}) \times \prod_{i\in I'\setminus\{j\}} v_i \longrightarrow \Gamma^1(c_{j,p}) \times \prod_{i\in I'\setminus\{j\}} v_i.$$

The analogous result holds for φ_q, which shows that these maps are compatible with the projection map $\mathrm{pr}_{I''}$ and hence with the specialization map. This proves that the diagram in item (2) of Definition 3.12 commutes, as we want.

The proposition follows from Steps 1–7 above. $\square$

3.4.3. *Proof of Theorem 3.15.*

Proof of Theorem 3.15. Let $Y \to S$ be an S-colored D-quasi-nodal morphism. Fix a total ordering on the set Ω from Definition 3.2. We construct $g\colon Y' \to Y$ as a sequence of blow-ups as follows. We start with the smallest element $\alpha \in \Omega$ and we blow-up (the proper transform of) E_α repeatedly as long as $o_\alpha(Y) \geq 2$. After each such blow-up, we replace Ω by $\Omega' = \Omega \sqcup \{\beta\}$, where β is a new element that we define to be larger than all elements of Ω. By Proposition 3.19, $o_\alpha(Y') = o_\beta(Y') = o_\alpha(Y) - 1$, while $o_\gamma(Y') = o_\gamma(Y)$ for all $\gamma \in \Omega \setminus \{\alpha\}$. We repeat this process until $o_\alpha(Y) \leq 1$, at which point we replace α with the next element in Ω. Note that the maximum of $o_\alpha(Y')$ with $\alpha \in \Omega'$ drops by 1 via the above process after we have run through all elements in the original set $\Omega \subset \Omega'$. Hence, the algorithm terminates and produces an S-colored D-quasi-nodal morphism $Y' \to S$ with $o_\alpha(Y') \leq 1$ for all $\alpha \in \Omega'$. So $Y' \to S$ is nearly D-nodal, see Definition 2.4.

At each step of the algorithm, we have constructed a collection of maps $\varphi = \{\varphi_p\}_{p \in B}$ of 1-skeleta satisfying hypotheses (1) and (2) of Definition 3.12. The composition of those admissible modifications that appear in Proposition 3.19 are again admissible (because $Y' \to B$ will be S-colored and D-quasi-nodal, and $\hat{\Gamma}^1(Y'_p) = \Gamma^1(Y'_p)$ for all $p \in B$). Proposition 3.19 thus reduces the theorem to the case treated in Proposition 3.20. $\square$

3.4.4. *Base change of D-nodal morphisms.* The following lemma is our original motivation for Definitions 3.1 and 3.2.

Lemma 3.21. *Let $(\bar{B}, \bar{D})$ be a pair of a smooth variety $\bar{B}$ and an snc divisor $\bar{D}$ with components $\bar{D}_s$, $s \in S$. Let $\bar{f}\colon \bar{Y} \to \bar{B}$ be a strict $\bar{D}$-nodal morphism. Let B be a regular variety and let $\tau\colon B \to \bar{B}$ be a morphism such that $D = \tau^{-1}(\bar{D})_{\mathrm{red}}$ is an snc divisor on B with components $\bar{D} = \bigcup \bar{D}_i$, $i \in I$. Then $Y \coloneqq \bar{Y} \times_{\bar{B}} B$ is canonically S-colored D-quasi-nodal over (B, D).*

Proof. Let $\bar{\Omega}_s$ be the indexing set such that $\bar{E}_\alpha$ with $\alpha \in \bar{\Omega}_s$ are the irreducible components of $\bar{Y} \times_{\bar{B}} \bar{D}_s$. Since $\bar{f}\colon \bar{Y} \to \bar{B}$ is $\bar{D}$-nodal, the local charts $c \in \mathcal{C}$ of $\bar{Y}$ are of the form $\{h_s = x_s y_s \mid s \in S'\}$ for some subset $S' \subset S$, where $h_s = 0$ is a local equation of $\bar{D}_s$. Since f is strict, there are no self-intersections of the divisors $\bar{E}_\alpha$ in these charts. Hence, for $\alpha \in \bar{\Omega}_s$, $\bar{E}_\alpha \cap c$ is either empty or it is cut out by (h_s, x_s) or (h_s, y_s).

Let $D = \bigcup D_i$, $i \in I$ be the components of D. We have $\tau^* \bar{D}_s = \sum_{i \in I} a_{si} D_i$ for non-negative integers a_{si}. In local charts, the function h_s then pulls back to the monomial $\tau^* h_s = \prod_i u_i^{a_{si}}$, where locally $D_i = \{u_i = 0\}$. This shows that f is D-quasi-nodal.

We define

$$\Omega_s := \left\{(i,\alpha) \in I \times \bar{\Omega}_s \mid a_{si} > 0\right\}.$$

For $(i,\alpha) \in \Omega_s$, we then define the effective Weil divisor $E_{i,\alpha}$ as the reduction of the intersection

$$(Y \times_{\bar{Y}} \bar{E}_\alpha) \cap (Y \times_B D_i).$$

From this it is clear that the divisors indexed by $\Omega := \bigsqcup_{s\in S} \bar{\Omega}_s$ have the properties stated in item (1) in Definition 3.2. Similarly, one checks that item (2) holds true, which concludes the proof of the lemma. □

3.4.5. *Bicoloring.*

Definition 3.22. Let B be a smooth variety and let $D \subset B$ be an snc divisor with components D_i, $i \in I$. Let $Y \to B$ be a morphism which is D-nodal, nearly D-nodal or D-semistable. Then for each $p \in B$ the 1-skeleton $\Gamma^1(Y_p)$ carries a unique I-coloring, given by the condition that an edge e has color i, if the corresponding node is a specialization of a node over the generic point of D_i.

Let $Y \to B$ be an S-colored D-quasi-nodal morphism. By Theorem 3.15, there is an admissible modification $Y' \to Y$ such that Y' is strict D-semistable. It follows that the graphs $\Gamma^1(Y'_p)$ carry an I-coloring. This induces a unique I-coloring on the refinement $\hat{\Gamma}^1(Y'_p)$. Hence, $\hat{\Gamma}^1(Y'_p)$ carries both an I-coloring and an S-coloring. We will say that an edge e has *bicolor* (i,s) if it has color i in the given I-coloring and color s in the given S-coloring.

We are now able to formulate the following consequence of the proof of Theorem 3.15, that we will need.

Corollary 3.23. *Let $(\bar{B},\bar{D})$ be a pair of a smooth variety $\bar{B}$ and an snc divisor $\bar{D}$, $\bar{D} = \bigcup_{s\in S} \bar{D}_s$. Let $\bar{Y} \to \bar{B}$ be a $\bar{D}$-nodal morphism and let $\tau\colon B \to \bar{B}$ be an alteration with B regular such that $D := \tau^{-1}(\bar{D})_{\mathrm{red}}$ is an snc divisor with components D_i, $i \in I$. Consider the base change $Y := \bar{Y} \times_B B$, which is naturally S-colored D-quasi-nodal over B by Lemma 3.21. Assume that for each $s \in S$ there is a component D_{i_s} of D with $\tau(D_{i_s}) = \bar{D}_s$ and such that $\tau\colon B \to \bar{B}$ is étale at the generic point of D_{i_s}. Choose a total ordering on Ω such that any element of $\bigsqcup_{s\in S} \Omega_{s,i_s}$ (see Remark 3.4) is larger than an element outside this subset. Let $(g\colon Y' \to Y, \varphi)$ be the admissible resolution associated to this ordering as in Theorem 3.15.*

Then, for $p \in D_{i_s}$, the following holds:

1. *Any edge of bicolor (i_s,s) of the refinement $\hat{\Gamma}^1(Y'_p)$ of $\Gamma^1(Y'_p)$ is an edge of $\Gamma^1(Y'_p)$.*
2. *The graph $\hat{\Gamma}^1(Y'_p)$ does not have any edge of bicolor (i_s,t) with $t \neq s$.*

(3) *The map of S-colored graphs $\varphi_p\colon \hat{\Gamma}^1(Y'_p) \to \Gamma^1(Y_p)$ does not contract any edge of bicolor (i_s, s).*

(4) *Any s-colored edge of $\Gamma^1(Y_p)$ is the image of some edge of bicolor (i_s, s).*

Proof. By our construction, $\varphi_p\colon \hat{\Gamma}^1(Y'_p) \to \Gamma^1(Y_p)$ restricts to a map $\hat{\Gamma}^1(c'_p) \to \Gamma^1(c_p)$ for each chart $c \in \mathcal{C}$ as in Definition 3.1. It thus suffices to prove the corollary in the case where $Y = c$ is a single chart such that $\Gamma^1(c_p)$ contains an edge of color s. Let c be such a chart and assume for simplicity that p corresponds to 0 in this chart. Then the normal form (3.1.1) decomposes into a fiber product $W \times_B Z$, where

$$W := \left\{u_{i_s} \textstyle\prod_{i\in I\setminus\{i_s\}} u_i^{a_{si}} = x_s y_s\right\} \qquad \text{and} \qquad Z := \left\{\textstyle\prod_{i\in I\setminus\{i_s\}} u_i^{a_{ti}} = x_t y_t \mid t \in S' \setminus \{s\}\right\},$$

for some subset $S' \subset S$. Moreover, for $p \in B$, the edges of $\Gamma^1(Z_p)$ have colors contained in $S' \setminus \{s\}$, while the unique edge of $\Gamma^1(W_p)$ has color s.

Recall that in the resolution algorithm from Theorem 3.15, we are first blowing up E_α with $\alpha \in \Omega$ and $o_\alpha(Y) \geq 2$ according to the order of Ω, starting with the smallest element. By our choice of order, this means that in the above chart c, we first blow up (u_i, x_s) or (u_i, y_s) with $i \neq i_s$, and also possibly some (u_i, x_t) or (u_i, y_t) with $t \neq s$, before we blow up (u_{i_s}, x_s) or (u_{i_s}, y_s). If we blow-up (u_i, x_s), then the respective morphism φ_p has the property that s-colored edges contained in the blow-up chart (3.4.2) are contracted, while the s-colored edges in (3.4.1) are not.

Since we are only interested in edges of bicolor (i_s, s), we deduce from the local charts in (3.4.1) and (3.4.2) that after $a_{s,i}$-many blow-ups of (u_i, x_s) or (u_i, y_s) with $i \neq i_s$ (and also several blow-ups of (u_i, x_t) or (u_i, y_t) for $t \neq s$), we may without loss of generality reduce to the case of a product $W \times Z$ of the form

$$W := \{u_{i_s} = x_s y_s\} \qquad \text{and} \qquad Z := \left\{\textstyle\prod_{i\in I\setminus\{i_s\}} u_i^{a_{ti}} = x_t y_t \mid t \in S' \setminus \{s\}\right\}.$$

Our resolution algorithm from Theorem 3.15, applied to this product, will turn $W \times Z$ into the product $W \times Z'$ for some resolution $Z' \to Z$ such that the s-colored edges of $\hat{\Gamma}^1(W_p \times Z'_p)$ are given by $\Gamma^1(W_p) \times \hat{\Gamma}^0(Z'_p)$. This implies items (1) and (2) in the corollary. Moreover, the map φ_p respects this product structure (cf. Step 4 in Proposition 3.19 and Step 6 in Proposition 3.20). In other words, each element of $\Gamma^1(W_0) \times \hat{\Gamma}^0(Z'_0)$ is of the form $e \times v$ for some vertex $v \in \hat{\Gamma}^0(Z'_0)$ and $\varphi_p(e \times v) = e \times \varphi_p(v)$, where φ_p also denotes the map $\hat{\Gamma}^1(Z'_0) \to \Gamma^1(Z_0)$. This proves items (3) and (4) and concludes the proof. $\square$

4. From algebraicity to quadratic splittings of matroids

4.1. **Setup.** Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$, $s \mapsto y_s$, for some free $\mathbb{Z}$-module U of rank $g = \operatorname{rank}(\underline{R})$. Let B be a smooth affine variety with distinguished point $0 \in B$. Let h_s, $s \in S$, be regular functions on B such that

$(h_s)_{s\in S}\colon B \to \mathbb{A}^S$ is étale and such that h_s restricts to the s-th coordinate function on a disc $\Delta^S \subset B$ centered around 0. Up to shrinking B if necessary, we may assume that the divisors $H_s := \{h_s = 0\} \subset B$ are irreducible. We refer to H_s as the *s-th coordinate hyperplane* on B and denote by $H := \{\prod_{s\in S} h_s = 0\}$ the union of the coordinate hyperplanes H_s and by $B^\star := B \setminus H$ the complement of H in B.

Definition 4.1. Let $\pi^\star\colon X^\star \to B^\star$ be a smooth projective family of g-dimensional principally polarized abelian varieties. We say that $\pi^\star$ is a *matroidal family* associated to the regular matroid $(\underline{R}, S)$ with integral realization $S \to U^*$, $s \mapsto y_s$, if the following holds, where $t_0 \in (\Delta^\star)^S$ is a base point:

1. The family $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ over the punctured polydisc $(\Delta^\star)^S \subset B^\star$ has unipotent monodromies about the coordinate hyperplanes.
2. There exists an isomorphism
$$U \xrightarrow{\cong} \mathrm{gr}_0^W H_1(X_{t_0}, \mathbb{Z}),$$
where $W_\bullet$ denotes the weight filtration of the limit mixed Hodge structure associated to the family $X_{(\Delta^\star)^S} \to (\Delta^\star)^S$, under which
3. for all $s \in S$ there is a positive integer d_s such that the bilinear form $d_s y_s^2$ on U is identified with the monodromy bilinear form (cf. Definition 2.12) on $\mathrm{gr}_0^W H_1(X_{t_0}, \mathbb{Z})$ induced by the monodromy about $h_s = 0$.

Moreover, by Lemma 2.2, the above notion depends only on the matroid $\underline{R}$ and not on the chosen realization.

Remark 4.2. Since $B \to \mathbb{A}^S$ is étale, the cardinality of S coincides with the dimension of $B^\star$. Note moreover that we ask in Definition 4.1 that the dimension g of the fiber X_{t_0} agrees with the rank of the matroid $\underline{R}$. In other words, the corresponding degeneration of abelian varieties is maximal. In the proof of our main results, such as Theorems 1.6 and 1.8, we use this condition only when invoking [EGFS25a, Theorem 7.1] in Proposition 4.5 below; we are confident that the latter results also hold in the general case where the fiber dimension may be larger than the dimension of $\mathrm{gr}_0^W H_1(X_{t_0}, \mathbb{Z})$, but we did not try to prove this and concentrated on the case of maximal degenerations for simplicity.

Remark 4.3. By [EGFS25a, Proposition 4.10], for any regular matroid $(\underline{R}, S)$ with integral realization $S \to U^*$, there exists a family of principally polarized abelian varieties $\pi^\star\colon X^\star \to B^\star$ which is a matroidal family associated to $\underline{R}$. Furthermore, $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ is uniquely determined by $\underline{R}$ and $(d_s)_{s\in S}$ up to an analytic deformation, see [EGFS25a, Remark 2.31].

Remark 4.4. Let d be a positive integer with $d_s \mid d$ for all s, where d_s are the positive integers from Definition 4.1 above. Up to possibly shrinking B and performing a base change that restricts to $(h_s)_{s\in S} \mapsto (h_s^{d/d_s})_{s\in S}$ on the polydisc Δ^S, we may assume that $d_s = d$ for all s.

Proposition 4.5. *Let $\pi^\star\colon X^\star \to B^\star$ be a matroidal family of principally polarized abelian varieties, associated to $(\underline{R}, S)$. Then, up to replacing B by an étale local neighborhood of 0, there is a flat projective morphism $\pi\colon X \to B$ of varieties, such that:*

(1) $X \times_B B^\star = X^\star$;
(2) $\pi\colon X \to B$ is an H-nodal morphism, where $H = \left\{\prod_{s\in S} h_s = 0\right\}$; if $d_s \geq 2$ for all s in item (3), then π is in fact strict H-nodal.

Proof. Note that flatness is a formal consequence of being H-nodal. The result follows by applying a suitable Mumford construction, see [EGFS25a, Theorem 7.1]. □

4.2. **Quadratic splittings of matroids.** Let G be an S-colored graph with edge set $E = \bigsqcup_{s\in S} E_s$. The choice of an orientation on each edge of E induces an inclusion $H_1(G,\Lambda) \subset \Lambda^E$. For $e \in E$, we denote by $x_e\colon \Lambda^E \to \Lambda$ the e-th coordinate function. According to our conventions, we denote the associated bilinear form by x_e^2 and we denote by the same symbol its restriction to $H_1(G,\Lambda)$. These bilinear forms do not depend on the choice of orientation, because changing the orientation of e changes x_e by a sign.

We have the following variant of Definition 1.7 from the introduction.

Definition 4.6. Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$, $s \mapsto y_s$. Let Λ be a ring and let d be a positive integer. A *quadratic Λ-splitting of level d* of $(\underline{R}, S)$ in a graph G is an S-coloring $E = \sqcup_{s\in S} E_s$ of the edges of G together with an embedding $U_\Lambda \hookrightarrow H_1(G,\Lambda)$ which induces a decomposition

$$H_1(G,\Lambda) = U_\Lambda \oplus U', \tag{4.2.1}$$

for some $U' \subset H_1(G,\Lambda)$, such that for all $s \in S$, the following holds for the bilinear form $Q_s \coloneqq \sum_{e\in E_s} x_e^2$ on $H_1(G,\Lambda)$:

(1) the decomposition (4.2.1) is orthogonal with respect to Q_s;
(2) the restriction of Q_s to U_Λ agrees with $d \cdot y_s^2$.

Remark 4.7. It follows from Lemma 2.2 that the existence of a quadratic Λ-splitting of $(\underline{R}, S)$ in a graph G does not depend on the choice of integral realization.

Remark 4.8. Definition 4.6 is a special case of a quadratic Λ-splitting of level d of $(\underline{R}, S)$ in the cographic matroid associated to G (see Definition 1.7); it corresponds to the special case where the $\sum_s a_{se} = 1$ for all edges $e \in E$. The following lemma shows that in fact both notions are equivalent. This implies for instance that the notion passes to deletions, as this is obvious for Definition 1.7.

Lemma 4.9. *Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$, $s \mapsto y_s$. Then $(\underline{R}, S)$ admits a quadratic Λ-splitting of level d in a cographic matroid $M^*(G)$ associated to a graph G (see Definition 1.7) if and only if it admits a quadratic Λ-splitting of level d in some graph (see Definition 4.6).*

Proof. One direction is obvious, see Remark 4.8. For the converse, consider the non-negative integers a_{se} from Definition 1.7. We may first divide each edge $e \in E$ with $\sum_s a_{se} \geq 1$ into a chain of $\sum_s a_{se}$ colored edges, such that precisely a_{se}-many of them have color s. This reduces us to the situation where $\sum_{s \in S} a_{se} \in \{0, 1\}$ for all $e \in E$. We may then define a partial S-coloring of G by declaring an edge $e \in E$ to be of color s if and only if $a_{se} = 1$. We thus obtain all the conditions from Definition 4.6 apart from the fact that the S-coloring of G is only a partial S-coloring. We refer to this condition as a *weak quadratic Λ-splitting of level d into the graph G.*

To prove the lemma, it suffices to reduce to the case where the aforementioned partial S-coloring is in fact an S-coloring, i.e. to the case where $\sum_{s \in S} a_{se} = 1$ for all $e \in E$. We do so by induction on the number of colorless edges of G. To this end, let $e \in E$ be a colorless edge and let $G \to G'$ be the contraction of e. If e is not a loop, then it can be extended to a spanning forest of G. It follows that $H_1(G, \Lambda) \cong H_1(G', \Lambda)$ and we obtain an induced weak Λ-splitting of $\underline{R}$ in G' which has fewer colorless edges than the one into G. If e is a loop, then $e \in U'$ because the linear forms y_s, $s \in S$, form a basis of U^*. It is then again easy to see that we obtain an induced weak Λ-splitting of $\underline{R}$ in G' which has fewer colorless edges than the one into G. This concludes the proof. □

The main result of this section is the following, where B and $B^\star = B \setminus H$ are as in Section 4.1.

Theorem 4.10. *Let $\pi^\star \colon X^\star \to B^\star$ be a matroidal family of principally polarized abelian varieties associated to a regular matroid $(\underline{R}, S)$ with integral realization $S \to U^*$, $s \mapsto y_s$, see Definition 4.1. Let ℓ be a prime and assume that an ℓ-prime multiple of the minimal curve class of the geometric generic fiber of π is represented by an algebraic curve. Let Λ be a $\mathbb{Z}_{(\ell)}$-algebra. Then there is a positive integer d and an S-colored graph G, such that the matroid $(\underline{R}, S)$ admits a quadratic Λ-splitting of level d in G.*

Remark 4.11. The proof will show that the integer d in the above theorem does not depend on Λ. Another way of seeing this fact is to note that the theorem follows formally from the case $\Lambda = \mathbb{Z}_{(\ell)}$ after applying $\otimes_{\mathbb{Z}_{(\ell)}} \Lambda$, and so the value of d that works for $\mathbb{Z}_{(\ell)}$ works in fact for any $\mathbb{Z}_{(\ell)}$-algebra Λ.

The remainder of this section is devoted to a proof of the above theorem. To this end, we will in this section denote by Λ an algebra over the localization $\mathbb{Z}_{(\ell)}$ of $\mathbb{Z}$. We further use Remark 4.4 to assume without loss of generality that

$$\text{there is an integer } d \geq 2 \text{ with } d_s = d \text{ for all } s \in S \text{ in Definition 4.1.} \tag{4.2.2}$$

That is, the s-th monodromy bilinear form associated to $\pi^\star\colon X^\star \to B^\star$ identifies to dy_s^2 for all $s \in S$. Since $d \geq 2$, we may by Proposition 4.5 choose a strict H-nodal extension $\pi\colon X \to B$ of $\pi^\star$.

4.3. **Base change and resolution.** By the assumption in Theorem 4.10, the geometric generic fiber of π contains a curve whose cohomology class is m times the minimal class for some integer m that is coprime to ℓ. While Proposition 4.5 allows us to freely perform base changes of B that are only ramified along H, such base changes will in general not suffice to descend this curve to the generic fiber of $X \to B$. Nonetheless, we are able to make the following reductions.

Lemma 4.12. *In order to prove Theorem 4.10, we may assume that there is an alteration*

$$\tau\colon B' \longrightarrow B$$

with base change $X' = X \times_B B' \to B'$ and the following properties:

1. *The generic fiber X'_η of $X' \to B'$ contains a finite number of integral curves $C'_j \subset X'_\eta$, each with a rational point in its smooth locus, such that $\sum_j [C'_j] \in H^{2g-2}(X'_{\bar\eta}, \mathbb{Z}_\ell)$ is an ℓ-prime multiple of the minimal class.*
2. *B' is regular and the reduction $D := \tau^{-1}(H)_{\mathrm{red}}$, where $H = \{\prod_{s\in S} h_s = 0\}$, is an snc divisor with components D_i, $i \in I$; the reduced preimage $\tau^{-1}(0)_{\mathrm{red}}$ is an snc divisor as well.*
3. *For each $s \in S$, there is a component D_{i_s} of D such that $\tau(D_{i_s}) = H_s$ and τ is étale at the generic point of D_{i_s}.*
4. *The morphism $X' \to B'$ is D-quasi-nodal and admits a natural S-coloring via Lemma 3.21. There is an admissible resolution $(g\colon X'' \to X', \varphi)$ as in Theorem 3.15 such that the additional properties from Corollary 3.23 hold true. That is, if D_{i_s} is as in (3) and $p \in D_{i_s}$, then the following hold:*
   - (a) *Any edge of bicolor (i_s, s) of the refinement $\hat\Gamma^1(X''_p)$ of $\Gamma^1(X''_p)$ is an edge of $\Gamma^1(X''_p)$, cf. Definition 3.12.*
   - (b) *The graph $\hat\Gamma^1(X''_p)$ does not have any edge of bicolor (i_s, t) with $t \neq s$.*
   - (c) *The map of S-colored graphs $\varphi_p\colon \hat\Gamma^1(X''_p) \to \Gamma^1(X'_p)$ does not contract any edge of bicolor (i_s, s).*
   - (d) *Any edge of $\Gamma^1(X'_p)$ of color s is in the image of some edge of bicolor (i_s, s).*

Proof. In the notation of Theorem 4.10, there is a finite field extension $L/\mathbb{C}(B)$, such that the base change X_L contains an effective curve whose cohomology class in the geometric generic fiber $X_{\bar L}$ is an ℓ-prime multiple of the minimal class. Up to enlarging L, we may assume that each component of the reduction of this curve admits an L-rational point in its smooth locus. A straightforward (and well-known) computation with ramification indices shows that we can find a sequence of alterations of normal varieties

$$B' \longrightarrow \widetilde{B} \longrightarrow B$$

and some integer $d' \geq 1$, such that $L \subset \mathbb{C}(B')$ and the following hold:

- $\widetilde{B} \to B$ is finite and restricts to the cover $(h_s)_{s\in S} \mapsto (h_s^{d'})_{s\in S}$ over $\Delta^S \subset B$;
- $B' \to \widetilde{B}$ is étale over the generic points of the components of $\tilde{H}_s$, where $\tilde{H}_s \subset \widetilde{B}$ denotes the reduction of the preimage of $H_s \subset B$ in $\widetilde{B}$.

Up to replacing d by dd' in (4.2.2), we may, by Proposition 4.5, assume without loss of generality that $B = \widetilde{B}$. By resolution of singularities, we thus may further assume that there is an alteration

$$\tau\colon B' \longrightarrow B$$

which satisfies items (1)–(3) in the lemma.

To prove item (4), we recall that $\pi\colon X \to B$ is strict H-nodal. By Lemma 3.21 (applied to $\bar{B} = B$, $\bar{D} = H$, and $\bar{Y} = X$), the base change $X' = X \times_B B'$ is thus canonically S-colored D-quasi-nodal, where $D = \tau^{-1}(H)_{\mathrm{red}}$, see Definition 3.2. In particular, we get an index set $\Omega = \sqcup_s \Omega_s$ and a collection of Weil divisors E_α for all $\alpha \in \Omega_s$. By Remark 3.4, we have a canonical partition $\Omega_s = \sqcup_{i\in I} \Omega_{s,i}$, where I is the index set of the components of D. By item (2), for each $s \in S$ there is a component D_{i_s} of D such that $\tau(D_{i_s}) = H_s$ and τ is étale at the generic point of D_{i_s}. We then pick a total ordering of Ω such that any element of $\bigcup_{s\in S} \Omega_{s,i_s}$ is larger than any element outside this subset. We apply Theorem 3.15 to this total ordering and get an admissible resolution $(g\colon X'' \to X', \varphi)$ of $X' \to B'$, which, by our choice of ordering, satisfies the conclusions of Corollary 3.23. $\square$

The remainder of this section is devoted to a proof of Theorem 4.10. To this end, we will from now on assume that (4.2.2) as well as items (1)–(4) in Lemma 4.12 hold true.

We summarize the morphisms from Lemma 4.12 in the following diagram:

$$\begin{tikzcd}
X'' \arrow[r, "g"] \arrow[dr] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& B' \arrow[r, "\tau"] & B.
\end{tikzcd}$$

4.4. **Bringing the curve into good position.**

Proposition 4.13. *There is a vector bundle $\mathcal{E}$ on X'', such that for $b \in B'$ general, the $(g-1)$-st Segre class $s_{g-1}(\mathcal{E}|_{X''_b})$ is an ℓ-prime multiple of the minimal class of $X''_b = X_{\tau(b)}$.*

Proof. As a consequence of item (1) in Lemma 4.12, there is an open subset $U' \subset B'$ with $X''_{U'} = X'_{U'}$ and a smooth projective family of curves $C' \to U'$, whose fibers may be disconnected but such that each component admits a section, together with a proper morphism $h\colon C' \to X''_{U'}$, such that $h_*[C'_b]$ is m times the minimal class of $X''_b = X'_b = X_{\tau(b)}$ for all $b \in U'$, where m is coprime to ℓ. The existence of the sections determines an Abel–Jacobi mapping $C' \to JC'$ and hence a morphism of abelian schemes $JC' \to X''_{U'}$, where JC' denotes the relative Jacobian of C' over U'. We apply Lemma 2.9 to the generic fiber of $C' \to U'$ and spread the result out to a neighborhood of the generic point of U'. Then, by that lemma, up to shrinking U', we can assume that there is a vector bundle $\mathcal{F}_{U'}$ on $X''_{U'}$ whose i-th Segre class restricts on a general fiber X''_b to $m^i \cdot \Theta^i/i!$ for all i. Here, Θ denotes the theta divisor class on X''_b. By [Har77, II, Exercise 5.15], there exists a coherent sheaf $\mathcal{F}$ on X'' that restricts to the vector bundle $\mathcal{F}_{U'}$ on $X''_{U'} \subset X''$. This coherent sheaf has then the property that, for general $b \in B'$,

$$s(\mathcal{F}|_{X''_b}) = e^{m\Theta}. \tag{4.4.1}$$

We pick a sufficiently ample line bundle L on X'', such that $c_1(L)$ is ℓ-divisible. Consider the natural map

$$\mathcal{E}^0 := H^0(X, \mathcal{F} \otimes L^n) \otimes L^{-n} \longrightarrow \mathcal{F},$$

which is surjective for $n \gg 0$. Applying the same construction to the kernel of the above map and repeating the process, we produce by Hilbert's syzygy theorem (which uses that X'' is regular) a finite locally free resolution

$$\mathcal{E}^\bullet := 0 \to \mathcal{E}^N \to \mathcal{E}^{N-1} \to \cdots \to \mathcal{E}^2 \to \mathcal{E}^1 \to \mathcal{E}^0 \to \mathcal{F} \to 0,$$

of $\mathcal{F}$, where $N = \dim X''$ and $\mathcal{E}^i$ is, for $i < N$, a direct sum of some tensor powers of L. Then the total Segre class of $\mathcal{F}$ satisfies

$$s(\mathcal{F}) = \prod_i s(\mathcal{E}^{2i}) \cdot c(\mathcal{E}^{2i-1}).$$

We let $\mathcal{E} := \mathcal{E}^N$. Since $c_1(L)$ is ℓ-divisible and $\mathcal{E}^i$ is, for $i < N$, a direct sum of some tensor powers of L, we deduce:

$$c(\mathcal{E}^i) \equiv s(\mathcal{E}^i) \equiv 1 \mod \ell \qquad \text{for } i < N.$$

Hence,

$$s(\mathcal{F}) \equiv \begin{cases} s(\mathcal{E}) \mod \ell & \text{if } N \text{ is even;} \\ c(\mathcal{E}) \mod \ell & \text{if } N \text{ is odd.} \end{cases}$$

By (4.4.1), and the fact that the total Chern class is the inverse of the total Segre class, we find that

$$s(\mathcal{E}|_{X''_b}) \equiv e^{\pm m\Theta} \mod \ell.$$

Since m is coprime to ℓ, it follows that $s_{g-1}(\mathcal{E}|_{X''_b})$ is an ℓ-prime multiple of the minimal class, for $b \in B''$ general, as desired. $\square$

Let $\mathcal{E}$ be the vector bundle from Proposition 4.13. We define

$$Y := \mathbb{P}(\mathcal{E}).$$

Note that $Y \to X''$ is a smooth morphism. In particular, $Y \to B'$ is strict D-semistable (hence flat), because the same holds for $X'' \to B'$. Let further L be a very ample line bundle on $\mathbb{P}(\mathcal{E})$, given by the tensor product of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ with an ℓ-divisible and sufficiently positive bundle, and consider a complete intersection

$$C \subset Y$$

of $r + g - 1$ many general elements of $|L|$, where $r = \mathrm{rk}(\mathcal{E}) - 1$. Note that C is regular by Bertini's theorem. It is not hard to see that $C \to B'$ is a flat family of curves, but we will only use the following more precise statement for a restriction of this morphism to a suitable open subset $B^\circ \subset B'$.

Lemma 4.14. *Let $P \subset B'$ be a finite set of closed points. Then there is a dense open subset $B^\circ \subset B'$ which contains P, such that the following holds:*

(1) The base change $C_{B^\circ} \to B^\circ$ is strict D°-semistable, where $D^\circ = D \cap B^\circ$. For $b \in B^\circ$, $C_b \subset Y_b$ meets each double locus of Y_b transversely, but misses any deeper stratum of Y_b, and induces a bijection on irreducible components. Moreover, the singularities of the morphism $C_{B^\circ} \to B^\circ$ are D°-nodal, given by transverse slices of the singularities of $Y_{B^\circ} \to B^\circ$.

(2) If $g \geq 3$, then for all $p \in B^\circ$, the natural map $\iota \colon C_b \to X''_b$ is a closed embedding.

(3) For general $b \in B^\circ$, $\iota_*[C_b]$ represents m-times the minimal class of $X''_b = X_{\tau(b)}$, for some positive integer m that is coprime to ℓ.

Proof. Item (1) follows by applying the relative Bertini theorem in the form of Lemma 2.6. Item (2) follows up to shrinking B° around P from the assumption that $g \geq 3$ and the well-known fact that a general complete intersection curve embeds via a smooth morphism whose target has dimension at least 3 into the target of the morphism. Finally, item (3) follows from the fact that the $(g-1)$-st Segre class of $\mathcal{E}$ restricts to an ℓ-prime multiple of the minimal class on the general fiber of $X'' \to B'$. $\square$

To fix notation, we make now the following

Definition 4.15. We fix a finite set $P \subset B^\circ \cap \tau^{-1}(0)$ of closed points which contains a point on each connected component of each non-empty open stratum D_J° of D which is contained in $\tau^{-1}(0)$. We then denote by $B^\circ \subset B'$ the open subset from Lemma 4.14 which contains P and such that items (1)–(3) in the lemma hold true.

Since $(g \colon X'' \to X', \varphi)$ is admissible, for each $p \in B'$ there is a canonical S-colored graph $\hat{\Gamma}^1(X''_p)$, given by a refinement of $\Gamma^1(X''_p)$, see Definition 3.12. Since $Y = \mathbb{P}(\mathcal{E}) \to$

X'' is smooth, we further have for all $p \in B'$ a canonical identification of dual complexes $\Gamma(Y_p) = \Gamma(X''_p)$ and associated graphs

$$\Gamma^1(Y_p) = \Gamma^1(X''_p).$$

Proposition 4.16. *The following holds for all $p \in B^\circ$.*

(1) The natural map of graphs $\Gamma(C_p) \to \Gamma^1(Y_p) = \Gamma^1(X''_p)$ is an isomorphism on vertices, surjective on edges and does not contract any edge; in other words, the map identifies certain multiple edges to a single edge.

(2) Let $\hat{\Gamma}(C_p)$ be the refinement of $\Gamma(C_p)$ induced via $\Gamma(C_p) \to \Gamma^1(X''_p)$ from the refinement $\hat{\Gamma}^1(X''_p)$ of $\Gamma^1(X''_p)$. Then $\hat{\Gamma}(C_p)$ carries a canonical (I, S)-bicoloring such that the natural map

$$\hat{\Gamma}(C_p) \longrightarrow \hat{\Gamma}^1(X''_p) \tag{4.4.2}$$

is a map of (I, S)-bicolored graphs, which is an isomorphism on the set of vertices, surjective on the set of edges and does not contract any edge.

(3) For $s \in S$, let $i_s \in I$ be the index with $\tau(D_{i_s}) = H_s$ from item (3) in Lemma 4.12. Then the following holds for $p \in B^\circ \cap D_{i_s}$:

(a) Any edge e of $\hat{\Gamma}(C_p)$ of bicolor (i_s, s) is an edge of $\Gamma(C_p)$ and, moreover, $C \to B^\circ$ is locally at the corresponding node of C_p given by the product of the normal form $\{u_{i_s} = x_s y_s\}$ with a smooth fibration, where u_{i_s} is a local equation for D_{i_s}.

(b) The graph $\hat{\Gamma}(C_p)$ has no edge of bicolor (i_s, t) with $t \neq s$.

(c) The composition $\hat{\Gamma}(C_p) \to \hat{\Gamma}^1(X''_p) \xrightarrow{\varphi_p} \Gamma^1(X'_p)$ is a map of S-colored graphs which does not contract any edge of bicolor (i_s, s) and such that any edge of $\Gamma^1(X'_p)$ of color s is the image of some edge of $\hat{\Gamma}(C_p)$ of bicolor (i_s, s).

Proof. Item (1) follows from item (1) in Lemma 4.14, because $\Gamma^1(Y_p) = \Gamma^1(X''_p)$ for all p, since $Y \to X''$ is a smooth morphism (it is a projective bundle). Item (2) is a straightforward consequence of item (1).

It remains to prove item (3). By item (1) in Lemma 4.14, $C_{B^\circ} \subset Y_{B^\circ}$ is a local complete intersection such that the singularities of $C_{B^\circ} \to B^\circ$ are given by transverse intersections of the singularities of $Y_{B^\circ} \to B^\circ$. Moreover, this intersection is in general position in the sense that the fibers of $C_{B^\circ} \to B^\circ$ meets the double locus of $Y_{B^\circ} \to B^\circ$ transversely but misses any deeper stratum, cf. Lemma 2.6. Using this, the claim in item (3) follows from item (4) in Lemma 4.12, together with the fact that $Y \to X''$ is a smooth morphism and so $\Gamma^1(Y_b) = \Gamma^1(X''_b)$ for all $b \in B^\circ$. $\square$

4.5. **A tree of compatible paths.** Consider $P \subset B^\circ$ from Definition 4.15. Recall that $P \subset \tau^{-1}(0)$ and $\tau^{-1}(0)_{\mathrm{red}}$ is an snc divisor on B by assumptions. Let

$$\psi_0 \colon \mathbb{T} \hookrightarrow \tau^{-1}(0) \tag{4.5.1}$$

be a topological embedding of a tree (i.e. of a finite connected graph with trivial first homology), such that:

- ψ_0 induces a bijection between the vertices of $\mathbb{T}$ and the set P;
- $\operatorname{im}(\psi_0) \subset B^\circ$;
- the interior of each edge e of $\mathbb{T}$ is embedded into some open stratum of D, i.e. $\psi_0(e \setminus \partial e) \subset D_J^\circ$ for some $J \subset I$.

Note that the normal bundle of D restricted to the graph $\psi_0(\mathbb{T})$ is trivial, as the latter is contractible. It follows that we can extend ψ_0 to a topological embedding

$$\psi \colon \mathbb{T} \times [0,1] \hookrightarrow B^\circ,$$

such that

- ψ_0 agrees with the restriction of ψ to $\mathbb{T} \times \{0\}$;
- $\psi(\mathbb{T} \times (0,1]) \subset B^\circ \setminus D$;
- for each $x \in \mathbb{T}$, $\psi(x \times [0,1])$ can be covered by an embedded polydisc $\Delta^S \subset B^\circ$ centered at $\psi_0(x)$.

We denote the restriction of ψ to $\mathbb{T} \times \{1\}$ by

$$\psi_{\mathrm{nb}} \colon \mathbb{T} \hookrightarrow B^\circ \setminus D. \tag{4.5.2}$$

For $x \in \mathbb{T}$ with $b := \psi_0(x)$, we call

$$t_b := \psi_{\mathrm{nb}}(x)$$

the *point nearby* to b. Moreover, $\psi(x, -)$ yields a path from $\psi_{\mathrm{nb}}(x)$ to $\psi_0(x)$.

Pick further a base point $t \in B'$ which lies above the base point $t_0 \in (\Delta^\star)^S \subset B^\star$ and a path γ which connects t to a vertex v_0 of $\psi_{\mathrm{nb}}(\mathbb{T})$. For any point $b \in \operatorname{im}(\psi)$, we then pick a path that connects t to b by first traveling along γ to the point $v_0 \in \psi_{\mathrm{nb}}(\mathbb{T})$, then traveling from v_0 to b along any path in $\operatorname{im}(\psi)$, which hits $\operatorname{im}(\psi_0)$ at most at the end point b if $b \in \operatorname{im}(\psi_0)$. Note that any two such choices of paths are homotopic to each other, because $\mathbb{T}$ is contractible. For this reason we may and will in what follows choose convenient paths as above whenever necessary.

For $p \in P$, let $v_p \in \mathbb{T}$ be the vertex with $\psi_0(v_p) = p$. We let further $t_p := \psi_{\mathrm{nb}}(v_p)$ be the point nearby to p. With the above set-up we then get for all $p \in P$ canonical identifications

$$\operatorname{gr}_0^{W^p} H_1(X_t'', \mathbb{Z}) \xrightarrow{\cong} \operatorname{gr}_0^{W^p} H_1(X_{t_p}'', \mathbb{Z}) \cong H_1(\Gamma(X_p''), \mathbb{Z}), \tag{4.5.3}$$

where $W^p_\bullet$ denotes the weight filtration on $H_1(X''_t, \mathbb{Z})$ (resp. $H_1(X''_{t_p}, \mathbb{Z})$) induced by the local monodromies at p and the given path from t to p (resp. t_p to p). The second isomorphism is detailed for instance in [EGFS25a, Proposition 5.10]; this uses the aforementioned embedded disc $\Delta^S \subset B$, centered at p and containing the point t_p. Similarly, we get canonical isomorphisms

$$\mathrm{gr}_0^{W^p} H_1(C_t, \mathbb{Z}) \xrightarrow{\cong} \mathrm{gr}_0^{W^p} H_1(C_{t_p}, \mathbb{Z}) \cong H_1(\Gamma(C_p), \mathbb{Z}). \tag{4.5.4}$$

Let e be an edge of $\psi_0(\mathbb{T})$ with end points p, q. There is some subset $J \subset I$ such that the interior of e and one of its end points, say p, lies in the open stratum D_J°. Then p specializes to q along the path given by the edge e of $\psi_0(\mathbb{T})$, as in Definition 3.11.

Lemma 4.17. *Let $p, q \in P$ such that p specializes to q in the above sense. Then there are natural commutative diagrams*

$$\begin{tikzcd}
\mathrm{gr}_0^{W^q} H_1(C_t, \mathbb{Z}) \arrow[r, "\cong"] \arrow[d] & H_1(\Gamma(C_q), \mathbb{Z}) \arrow[d, "\mathrm{sp}_*"] \\
\mathrm{gr}_0^{W^p} H_1(C_t, \mathbb{Z}) \arrow[r, "\cong"] & H_1(\Gamma(C_p), \mathbb{Z})
\end{tikzcd} \tag{4.5.5}$$

and

$$\begin{tikzcd}
\mathrm{gr}_0^{W^q} H_1(X''_t, \mathbb{Z}) \arrow[r, "\cong"] \arrow[d] & H_1(\Gamma(X''_q), \mathbb{Z}) = H_1(\Gamma(X_0), \mathbb{Z}) \arrow[d, "="] \\
\mathrm{gr}_0^{W^p} H_1(X''_t, \mathbb{Z}) \arrow[r, "\cong"] & H_1(\Gamma(X''_p), \mathbb{Z}) = H_1(\Gamma(X_0), \mathbb{Z})
\end{tikzcd} \tag{4.5.6}$$

where all horizontal maps are isomorphisms. The vertical arrows on the left are the natural quotient maps, induced by the inclusion $W^q_{-1} \subset W^p_{-1}$. The right most vertical arrow in (4.5.6) is induced by the fact that $\Gamma(X''_p)$ and $\Gamma(X''_q)$ are refinements of $\Gamma(X'_p) = \Gamma(X_0)$ and $\Gamma(X'_q) = \Gamma(X_0)$, respectively, see item (1) in Definition 3.12.

Moreover, the natural map $C_{B^\circ} \to X''_{B^\circ}$ (see Lemma 4.14) induces canonical maps from (4.5.5) to (4.5.6) such that the resulting cube of maps commutes.

Proof. The horizontal isomorphisms in diagrams (4.5.5) and (4.5.6) are the isomorphisms from (4.5.4) and (4.5.3), respectively, cf. [EGFS25a, Proposition 5.10]. Since $p, q \in \tau^{-1}(0)$, we have $X'_p = X_0$ and $X'_q = X_0$; all maps in (4.5.6) are isomorphisms and the commutativity is clear. To prove the commutativity of (4.5.5), recall from [EGFS25a, Propositions 5.9 and 5.10] that the second isomorphism in (4.5.5) is induced by a natural map $H_1(C_{t_p}, \mathbb{Z}) \to H_1(\Gamma(C_p), \mathbb{Z})$, which contracts all cycles that are invariant by the local monodromies at p. In other words, the vanishing cycles as well as the cycles that specialize in $H_1(C_p, \mathbb{Z})$ to cycles supported on the smooth locus of C_p are contracted.

Using this description together with the description of sp in Section 3.3 and our choice of compatible paths, the commutativity of (4.5.5) is easily verified.

Finally, the naturality of the maps in question implies that the canonical maps from (4.5.5) to (4.5.6) that are induced by the natural map $C_{B^\circ} \to X''_{B^\circ}$ make the resulting cube of maps commutative. $\square$

4.6. **1-skeletal splitting.** We have a natural surjective map

$$H_1(\Gamma^1(X_0), \Lambda) \longrightarrow H_1(\Gamma(X_0), \Lambda), \tag{4.6.1}$$

which is induced by the inclusion of the 1-skeleton $\Gamma^1(X_0)$ into the entire dual complex $\Gamma(X_0)$. We will show in this subsection that our set-up induces a canonical splitting of the above map, which we refer to as 1-*skeletal splitting*.

Recall from Definition 4.1 that we have specified an isomorphism

$$U \cong \mathrm{gr}_0^W H_1(X_{t_0}, \mathbb{Z}) = \mathrm{gr}_0^{W^p} H_1(X''_t, \mathbb{Z}),$$

where $t_0 \in B$ is the base point in $(\Delta^\star)^S \subset B^\star$, $t \in B'$ is a point with $\tau(t) = t_0$ and $W^p_\bullet$ denotes the weight filtration with respect to a point $p \in P$ and $\mathrm{gr}_0^{W^p} H_1(X''_t, \mathbb{Z})$ is independent of the choice of p by Lemma 4.17. By Lemma 4.17, we also get a natural isomorphism

$$U \cong H_1(\Gamma(X_0), \mathbb{Z}). \tag{4.6.2}$$

Consider the natural map $JC_{B^\circ} \to X''_{B^\circ}$, where JC_{B° denotes the relative Jacobian of C_{B° over B° and recall that $C_b \subset X''_b$ represents an ℓ-prime multiple of the minimal class for $b \in B^\circ$ general, see Lemma 4.14. Since ℓ-prime integers are invertible in Λ, Lemma 2.13 implies that for each $p \in P$, the natural map

$$\mathrm{gr}_0^{W^p} H_1(C_t, \Lambda) \longrightarrow \mathrm{gr}_0^{W^p} H_1(X''_t, \Lambda), \tag{4.6.3}$$

induced by the map $C_t \to X''_t$, admits a canonical splitting, induced by the map $X''_t \to JC_t$ that is dual to $JC_t \to X''_t$ with respect to the given principal polarizations (note that $X''_t = X_{t_0}$ is an abelian variety because $t_0 \notin H$ and g is an isomorphism away from $D = \tau^{-1}(H)_{\mathrm{red}}$, see Definition 3.12). We denote this canonical splitting of (4.6.3) by

$$\phi_{C_t}^{W^p} \colon U_\Lambda \longrightarrow \mathrm{gr}_0^{W^p} H_1(C_t, \Lambda). \tag{4.6.4}$$

Via the isomorphisms in Lemma 4.17, this yields a map

$$\phi_{C_p} \colon U_\Lambda \longrightarrow H_1(\Gamma(C_p), \Lambda), \tag{4.6.5}$$

uniquely determined by the topological embedding ψ, which splits the natural map

$$H_1(\Gamma(C_p), \Lambda) \longrightarrow H_1(\Gamma(X''_p), \Lambda) = H_1(\Gamma(X_0), \Lambda) = U_\Lambda. \tag{4.6.6}$$

Lemma 4.18. *Let $p \in P$ and $i \in I$ such that $p \in D_i$. Let $\hat{Q}_i$ be the monodromy bilinear form on $H_1(\Gamma(C_p), \Lambda) = \mathrm{gr}_0^{W^p} H_1(C_t, \Lambda)$ associated to the monodromy about D_i locally at p. Then the following holds:*

(1) the pullback $\phi_{C_p}^* \hat{Q}_i$ is the bilinear form on

$$U_\Lambda = \mathrm{gr}_0^W H_1(X_0, \Lambda) = \mathrm{gr}_0^W H_1(X'_p, \Lambda) = \mathrm{gr}_0^{W^p} H_1(X''_t, \Lambda),$$

which corresponds to m-times the monodromy bilinear form on $H_1(X''_t, \Lambda)$ associated to the monodromy about D_i locally at p (see Definition 2.12). (Here, m is the integer such that $[C_t] = m \cdot [\Theta]^{g-1}/(g-1)!$, see item (3) in Lemma 4.14.)

(2) the decomposition

$$H_1(\Gamma(C_p), \Lambda) = \phi_{C_p}(U_\Lambda) \oplus \ker(H_1(\Gamma(C_p), \Lambda) \to H_1(\Gamma(X''_p), \Lambda))$$

is orthogonal with respect to $\hat{Q}_i$.

Proof. This follows from Lemma 2.13 by applying $- \otimes_{\mathbb{Z}_\ell} \Lambda$. □

The main result of this subsection is the following

Proposition 4.19. *The choice of the topological embedding ψ from Section 4.5 induces a canonical map*

$$\phi \colon U_\Lambda \longrightarrow H_1(\Gamma^1(X_0), \Lambda)$$

which splits (4.6.1), (cf. (4.6.2)). Moreover, for each $p \in P$, ϕ agrees with the following composition

$$U_\Lambda \xrightarrow{\phi_{C_p}} H_1(\Gamma(C_p), \Lambda) \xrightarrow{\iota_*} H_1(\Gamma^1(X''_p), \Lambda) \longrightarrow H_1(\Gamma^1(X_0), \Lambda), \tag{4.6.7}$$

where ι_* is induced by the natural map $\iota \colon C_p \to X''_p$ from Lemma 4.14 and where the last arrow is the canonical map

$$H_1(\Gamma^1(X''_p), \Lambda) = H_1(\hat{\Gamma}^1(X''_p), \Lambda) \xrightarrow{\varphi_{p*}} H_1(\Gamma^1(X'_p), \Lambda) = H_1(\Gamma^1(X_0), \Lambda),$$

induced by the data φ of the admissible modification $X'' \to X'$ (see Definition 3.12).

Proof. For $p \in P$, let

$$\bar{\phi}_{C_p} \colon U_\Lambda \longrightarrow H_1(\Gamma^1(X_0), \Lambda)$$

be the composition of the maps in (4.6.7). To prove the proposition, we will show that this map defines a splitting which is independent of the choice of $p \in P$.

Let us first show that we get a splitting. To prove this, consider the following diagram, in which homology is taken with coefficients in Λ:

$$\begin{tikzcd}
H_1(\Gamma(X_0)) \arrow[r, "\phi_{C_p}"] \arrow[d, "\phi_{C_p}"] & H_1(\Gamma(C_p)) \arrow[r, "\iota_*"] & H_1(\Gamma^1(X_p'')) \arrow[r, "\varphi_{p*}"] \arrow[d] & H_1(\Gamma^1(X_p')) \arrow[r, "="] \arrow[d] & H_1(\Gamma^1(X_0)) \arrow[d] \\
H_1(\Gamma(C_p)) \arrow[r, "\iota_*"] & H_1(\Gamma^1(X_p'')) \arrow[r] & H_1(\Gamma(X_p'')) & H_1(\Gamma(X_p')) \arrow[l, "\cong"'] \arrow[r, "="] & H_1(\Gamma(X_0)).
\end{tikzcd}$$

The two squares on the left and right commute for obvious reasons, and the square in the middle commutes because $X'' \to X'$ is an admissible modification, see item (1) in Definition 3.12. Hence the outer square commutes. Moreover, the composition $H_1(\Gamma(X_0)) \to H_1(\Gamma^1(X_0))$ on the top row of this diagram is, by definition, the map $\bar{\phi}_{C_p}$, and the composition $H_1(\Gamma(C_p)) \to H_1(\Gamma(X_0))$ on the bottom row is the map (4.6.6). Since ϕ_{C_p} splits the map (4.6.6), precomposing the bottom row with ϕ_{C_p} yields the identity. Hence, precomposing the map $H_1(\Gamma^1(X_0)) \to H_1(\Gamma(X_0))$ with the top row is the identity, so that $\overline{\phi}_{C_p}$ splits (4.6.1), as claimed.

It remains to show that this splitting is independent from the choice of $p \in P$. Recall from Section 4.5 the construction of the tree $\psi_0(\mathbb{T}) \subset B^\circ \cap \tau^{-1}(0)$ whose set of vertices is P, and the choice of a tree of nearby points $\psi_{\mathrm{nb}}(\mathbb{T}) \subset B' \setminus D$. We further fixed a path from t to a vertex v_0 of $\psi_{\mathrm{nb}}(\mathbb{T})$ and used it to connect t to any other point of $\psi(\mathbb{T} \times [0,1])$ via the composition of γ with a path in $\psi([0,1] \times \mathbb{T})$, which is unique up to homotopy because the latter is contractible. We have used the corresponding paths to identify via Lemma 4.17 certain weight graded quotients of the homology above t with the homology of dual complexes above the points $p \in P$. Our choices have been made compatibly, in the sense that if $p, q \in P$ such that p specializes to q, then the maps induced by specializing from p to q are compatible, see Lemma 4.17.

Since $\mathbb{T}$ is a tree and P is its set of vertices, it then suffices to prove that if we have points $p, q \in P$ such that p specializes to q, the splitting induced by $\overline{\phi}_{C_p}$ agrees with the one induced by $\overline{\phi}_{C_q}$. Since $X'' \to X'$ is admissible, item (2) in Definition 3.12 implies that the following diagram commutes:

$$\begin{tikzcd}
H_1(\hat{\Gamma}^1(X_p''), \Lambda) = H_1(\Gamma^1(X_p''), \Lambda) \arrow[d, "\varphi_{p*}"] & H_1(\hat{\Gamma}^1(X_q''), \Lambda) = H_1(\Gamma^1(X_q''), \Lambda) \arrow[l, "\mathrm{sp}_*"'] \arrow[d, "\varphi_{q*}"] \\
H_1(\Gamma^1(X_p'), \Lambda) = H_1(\Gamma^1(X_0), \Lambda) & H_1(\Gamma^1(X_q'), \Lambda) = H_1(\Gamma^1(X_0), \Lambda). \arrow[l, "\mathrm{sp}_*"']
\end{tikzcd}$$

Moreover, the composition $\mathrm{sp}_* \circ \phi_{C_q}$ of the map $\phi_{C_q} \colon H_1(\Gamma(X_0), \Lambda) \to H_1(\Gamma(C_q), \Lambda)$ with the specialization map $\mathrm{sp}_* \colon H_1(\Gamma(C_q), \Lambda) \to H_1(\Gamma(C_p), \Lambda)$ is equal to the map $\phi_{C_p} \colon H_1(\Gamma(X_0), \Lambda) \to H_1(\Gamma(C_p), \Lambda)$. Together with the commutativity of the cubical diagram given by the natural map from (4.5.5) to (4.5.6) (see Lemma 4.17), it follows that each square in the following diagram commutes, where again homology is taken with coefficients in Λ:

$$\begin{tikzcd}
H_1(\Gamma(X_0)) \arrow[r, "\phi_{C_q}"] \arrow[d, equal] & H_1(\Gamma(C_q)) \arrow[r] \arrow[d, "\mathrm{sp}_*"] & H_1(\Gamma^1(X_q'')) \arrow[r, "\varphi_{q*}"] \arrow[d, "\mathrm{sp}_*"] & H_1(\Gamma^1(X_q')) \arrow[r, "="] \arrow[d, "\mathrm{sp}_*"] & H_1(\Gamma^1(X_0)) \arrow[d, equal] \\
H_1(\Gamma(X_0)) \arrow[r, "\phi_{C_p}"] & H_1(\Gamma(C_p)) \arrow[r] & H_1(\Gamma^1(X_p'')) \arrow[r, "\varphi_{p*}"] & H_1(\Gamma^1(X_p')) \arrow[r, "="] & H_1(\Gamma^1(X_0)).
\end{tikzcd}$$

As the composition on the top row equals $\bar{\phi}_{C_q}$ and the composition on the bottom row equals $\bar{\phi}_{C_p}$, we deduce that the splittings $\bar{\phi}_{C_q}$ and $\bar{\phi}_{C_p}$ agree with each other. This concludes the proof of the proposition. ☐

4.7. **Edge multiplication.** For each $s \in S$, recall from item (3) in Lemma 4.12 the component D_{i_s} of D with $\tau(D_{i_s}) = H_s$ such that $\tau\colon B' \to B$ is étale at the generic point of D_{i_s}. Since $P \subset \tau^{-1}(0)$ contains a point of each open stratum of D contained in $\tau^{-1}(0)$, we can, for each $s \in S$, pick a point $p_s \in P$ such that $p_s \in D_{i_s}$.

Definition 4.20. Let $s \in S$ and consider the refinement $\hat{\Gamma}(C_{p_s})$ of $\Gamma(C_{p_s})$ from Proposition 4.16. The *monodromy bilinear form* $\hat{Q}_s$ on $H_1(\hat{\Gamma}(C_{p_s}), \Lambda) = H_1(\Gamma(C_{p_s}), \Lambda)$ is the bilinear form that is induced by the local monodromy at p_s around the divisor D_{i_s}, see Definition 2.12.

Lemma 4.21. *Consider the refinement $\hat{\Gamma}(C_{p_s})$ of $\Gamma(C_{p_s})$ and let $E_{i_s,s}$ be the set of edges of $\hat{\Gamma}^1(C_{p_s})$ that have bicolor (i_s, s) (see item (2) in Proposition 4.16). Then each edge $e \in E_{i_s,s}$ is an edge of $\Gamma(C_{p_s})$ and we have*

$$\hat{Q}_s = \sum_{e \in E_{i_s,s}} x_e^2.$$

Proof. By item (3a) in Proposition 4.16, any edge of $\hat{\Gamma}(C_{p_s})$ of bicolor (i_s, s) is an edge of $\Gamma(C_{p_s})$. Moreover, if u_{i_s} denotes a local equation for D_{i_s}, the corresponding singularity of the map $C_{B^\circ} \to B^\circ$ is locally given by the product of $\{u_{i_s} = x_s y_s\}$ with a smooth fibration. So by the Picard–Lefschetz formula, each edge e of bicolor (i_s, s) contributes to Q_s with the summand x_e^2. Moreover, by item (3b) in Proposition 4.16, $\hat{\Gamma}^1(C_{p_s})$ has no edges of bicolor (i_s, t) with $t \in S \setminus \{s\}$. The claimed description of $\hat{Q}_s$ then follows from the fact that only edges of I-color i_s contribute to $\hat{Q}_s$, cf. Definition 3.22. ☐

Consider the map of S-colored graphs

$$\varphi_s\colon \hat{\Gamma}^1(C_{p_s}) \longrightarrow \Gamma^1(X'_{p_s}) = \Gamma^1(X_0)$$

given as the composition

$$\hat{\Gamma}^1(C_{p_s}) \longrightarrow \hat{\Gamma}^1(X''_{p_s}) \xrightarrow{\varphi_{p_s}} \Gamma^1(X'_{p_s}) = \Gamma^1(X_0),$$

where the first map comes from item (2) in Proposition 4.16. By item (3c) in Proposition 4.16, any edge of $\Gamma^1(X_0)$ of color s is the image of an edge of bicolor (i_s, s). Moreover, no edge of bicolor (i_s, s) is contracted. For any edge e of $\Gamma^1(X_0)$ which has color s, we may then define $F(e)$ as the subset of edges of $E(\hat{\Gamma}^1(C_{p_s}))$, given by

$$F(e) \coloneqq \{e' \in E(\hat{\Gamma}^1(C_{p_s})) \mid \varphi_s(e') = e \ \text{ and } e' \text{ has bicolor } (i_s, s)\}. \tag{4.7.1}$$

Remark 4.22. The elements of $F(e)$ correspond bijectively to the nodes of the curve C_{p_s} that do not smooth along the divisor D_{i_s}; this description follows from the fact that $\hat{\Gamma}^1(C_{p_s})$ has no nodes of bicolor (i_s, t) with $t \neq s$ and any edge of bicolor (i_s, s) of the refinement $\hat{\Gamma}^1(C_{p_s})$ is in fact an edge of $\Gamma^1(C_{p_s})$, see item (3) in Proposition 4.16.

Definition 4.23. For $s \in S$, we define the S-colored graph $\Gamma^1_s(X_0)$ to be the unique S-colored graph with a morphism $\pi_s \colon \Gamma^1_s(X_0) \to \Gamma^1(X_0)$, which is an isomorphism on vertices and an isomorphism on edges of color $t \in S \setminus \{s\}$, such that in addition, for any s-colored edge e of $\Gamma^1(X_0)$, we have an identification of $\pi_s^{-1}(e)$ with the set $F(e)$.

By construction, the natural map of S-colored graphs $\varphi_s \colon \hat{\Gamma}(C_{p_s}) \to \Gamma^1(X_0)$ factors naturally through a map of S-colored graphs

$$f_s \colon \hat{\Gamma}(C_{p_s}) \longrightarrow \Gamma^1_s(X_0),$$

such that $\varphi_s = \pi_s \circ f_s$.

Definition 4.24. For $s \in S$, let $E_s(\Gamma^1_s(X_0))$ denote the set of edges of $\Gamma^1_s(X_0)$ of color s. Then we define the *diagonal bilinear form*

$$Q_s \coloneqq \sum_{e \in E_s(\Gamma^1_s(X_0))} x_e^2$$

on $C_1(\Gamma^1_s(X_0), \Lambda)$ and we use the same symbol for its restriction to $H_1(\Gamma^1_s(X_0), \Lambda)$.

Lemma 4.25. *The map $f_s \colon \hat{\Gamma}(C_{p_s}) \to \Gamma^1_s(X_0)$ induces an isomorphism between the edges of bicolor (i_s, s) of $\hat{\Gamma}(C_{p_s})$ and the set of edges of color s of $\Gamma^1_s(X_0)$. In particular, Q_s pulls back to $\hat{Q}_s$ under $f_{s*} \colon H_1(\hat{\Gamma}(C_{p_s}), \Lambda) \to H_1(\Gamma^1_s(X_0), \Lambda)$.*

Proof. The fact that f_s induces an isomorphism between the edges of bicolor (i_s, s) of $\hat{\Gamma}(C_{p_s})$ and the set of edges of color s of $\Gamma^1_s(X_0)$ follows directly from the construction. The second claim is an immediate consequence of this and the description of $\hat{Q}_s$ given in Lemma 4.21. $\square$

Lemma 4.26. *Let $s \in S$. Then the natural composition*

$$H_1(\Gamma^1_s(X_0), \Lambda) \xrightarrow{\pi_{s*}} H_1(\Gamma^1(X_0), \Lambda) \longrightarrow H_1(\Gamma(X_0), \Lambda) = U_\Lambda \tag{4.7.2}$$

admits a canonical splitting

$$\phi_s \colon U_\Lambda \longrightarrow H_1(\Gamma^1_s(X_0), \Lambda),$$

given by $\phi_s = f_{s*} \circ \phi_{C_{p_s}}$. *Moreover:*

(1) *This splitting is orthogonal with respect to the bilinear form* Q_s *on* $H_1(\Gamma^1_s(X_0), \Lambda)$.
(2) $\phi_s^* Q_s = m \cdot d \cdot y_s^2$, *where* y_s^2 *is the rank one bilinear form associated to the realization* $S \to U^*$, $s \mapsto y_s$.

Proof. Recall that $\phi_{C_{p_s}} \colon U_\Lambda \to H_1(\Gamma(C_{p_s}), \Lambda)$ splits the natural map

$$H_1(\hat{\Gamma}(C_{p_s}), \Lambda) = H_1(\Gamma(C_{p_s}), \Lambda) \longrightarrow H_1(\Gamma(X_0), \Lambda),$$

see (4.6.5) and (4.6.6). The latter map factors as the composition of f_{s*} with the composition in (4.7.2). The first result of the lemma follows.

Item (1) then follows from the following facts: we have $(f_{s*})^* Q_s = \hat{Q}_s$ (see Lemma 4.25), the splitting $\phi_{C_{p_s}}$ is orthogonal with respect to $\hat{Q}_s$ (see Lemma 4.18), and the summands in the decomposition of $H_1(\hat{\Gamma}(C_{p_s}), \Lambda)$ surject onto the corresponding summands in the decomposition of $H_1(\Gamma^1_s(X_0), \Lambda)$.

To prove item (2), note that $\phi_s^* Q_s = \phi_{C_{p_s}}^* \hat{Q}_s$, and $\phi_{C_{p_s}}^* \hat{Q}_s$ is by Lemma 4.18 m-times the bilinear form on

$$U_\Lambda = \mathrm{gr}_0^W H_1(X_0, \Lambda) = \mathrm{gr}_0^{W^{p_s}} H_1(X_t, \Lambda),$$

associated to the local monodromy around the divisor D_{i_s} in B'. Since $\tau \colon B' \to B$ is étale at the generic point of D_{i_s}, this agrees via the first identity above with the local monodromy about the divisor H_s in B. Since $X^\star \to B^\star$ is a matroidal family associated to $(\underline{R}, S)$ with integral realization $S \to U^*$, $s \mapsto y_s$, item (3) in Definition 4.1 together with (4.2.2) implies that this bilinear form equals $d \cdot y_s^2$. Hence, we have $\phi_s^* Q_s = \phi_{C_{p_s}}^* \hat{Q}_s = m \cdot d \cdot y_s^2$, which proves item (2). $\square$

4.8. **Construction of G and proof of Theorem 4.10.**

Definition 4.27. We let G be the S-colored graph, given as the fiber product of the graphs $\Gamma^1_s(X_0)$, $s \in S$, over the graph $\Gamma^1(X_0)$.

Remark 4.28. The graph G can explicitly be described as the unique S-colored graph together with morphisms of S-colored graphs

$$\mathrm{pr}_s \colon G \to \Gamma^1_s(X_0) \qquad \text{and} \qquad g_s = \pi_s \circ \mathrm{pr}_s \colon G \to \Gamma^1(X_0),$$

which are isomorphisms on the set of vertices, such that, in addition, for any edge e of $\Gamma^1(X_0)$ of color s, the set $g_s^{-1}(e)$ is canonically identified with the set $F(e)$ defined in (4.7.1), consisting of edges of $\hat{\Gamma}(C_{p_s})$ of bicolor (i_s, s) which map to e via φ_s.

For $s \in S$, consider the natural composition

$$H_1(G, \Lambda) \xrightarrow{\mathrm{pr}_{s*}} H_1(\Gamma^1_s(X_0), \Lambda) \xrightarrow{\pi_{s*}} H_1(\Gamma^1(X_0), \Lambda) \longrightarrow H_1(\Gamma(X_0), \Lambda) \tag{4.8.1}$$

and note that this composition does not depend on s.

Proposition 4.29. *There is a canonical splitting*

$$\phi_G \colon U_\Lambda \longrightarrow H_1(G, \Lambda)$$

of (4.8.1) *such that for all $s \in S$ we have $\mathrm{pr}_{s*} \circ \phi_G = \phi_s$, where ϕ_s is the splitting from* (4.26)*.*

Proof. In order to define ϕ_G, we will first define a map

$$\phi_G \colon U_\Lambda \longrightarrow C_1(G, \Lambda)$$

and then show that the image of this map lies in $H_1(G, \Lambda)$.

Let $\alpha \in U_\Lambda$. We then define $\phi_G(\alpha) \in C_1(G, \Lambda)$ as the sum

$$\phi_G(\alpha) := \sum_{s \in S} c_s(\alpha),$$

where $c_s(\alpha)$ denotes the linear combination of oriented edges of color s, given by the s-colored part of $\phi_s(\alpha) \in H_1(\Gamma^1_s(X_0), \Lambda)$. This uses that there is a canonical isomorphism between the edges of color s of G and those of $\Gamma^1_s(X_0)$. In order to show that the above sum $\sum_{s \in S} c_s(\alpha)$ is closed, we first note that its pushforward to $C_1(\Gamma^1(X_0), \Lambda)$ equals

$$\sum_{s \in S} \pi_{s*}(\mathrm{pr}_{s*}(c_s(\alpha))) \in C_1(\Gamma^1(X_0), \Lambda). \tag{4.8.2}$$

We claim that the class (4.8.2) agrees with the 1-cycle $\phi(\alpha)$, where ϕ is the splitting from Proposition 4.19. To prove this, it suffices, for each $s \in S$, to check that the s-colored part of this 1-cycle agrees with the s-colored part of $\phi(\alpha)$, cf. Section 2.1.3. But the s-colored part of the above sum is nothing but $\pi_{s*}(\mathrm{pr}_{s*}(c_s(\alpha)))$. Hence, it suffices to show, for the s-colored part $c_s(\alpha) \in C_1(\Gamma^1_s(X_0), \Lambda)$ of the element $\phi_s(\alpha) \in H_1(\Gamma^1_s(X_0), \Lambda)$, that $\pi_{s*}(c_s(\alpha)) \in C_1(\Gamma^1(X_0), \Lambda)$ equals the s-colored part of $\phi(\alpha) \in H_1(\Gamma^1(X_0), \Lambda)$. For this, it suffices in turn to show that $\pi_{s*}(\phi_s(\alpha)) = \phi(\alpha) \in H_1(\Gamma^1(X_0), \Lambda)$, which follows from

$$\pi_{s*} \circ \phi_s = \pi_{s*} \circ f_{s*} \circ \phi_{C_{p_s}} = \varphi_{s*} \circ \phi_{C_{p_s}} = \phi.$$

We conclude that

$$\sum_{s \in S} \pi_{s*}(\mathrm{pr}_{s*}(c_s(\alpha))) = \phi(\alpha),$$

hence $\sum_{s\in S}\pi_{s*}(\mathrm{pr}_{s*}(c_s(\alpha)))$ is closed. Since $\pi_s\circ\mathrm{pr}_s\colon G\to\Gamma^1(X_0)$ is an isomorphism on vertices, we then deduce that $\sum_{s\in S}c_s(\alpha)$ must be closed, because its pushforward to $\Gamma^1(X_0)$ is closed. Hence,

$$\phi_G(\alpha)\coloneqq\sum_{s\in S}c_s(\alpha)\in H_1(G,\Lambda)$$

is well-defined. By construction, $c_s(\alpha)$ depends linearly on α and so ϕ_G is linear. The composition of ϕ_G with the natural map

$$H_1(G,\Lambda)\longrightarrow H_1(\Gamma^1(X_0),\Lambda)$$

agrees with ϕ from Proposition 4.19. Hence, ϕ_G is a splitting of (4.8.1), as desired. The equality $\mathrm{pr}_{s*}\circ\phi_G=\phi_s$ follows easily from the construction. □

For $s\in S$, let E_s denote the set of edges of G of color s. By construction, the edges of color s of G are canonically isomorphic to the edges of $\Gamma^1_s(X_0)$ of color s. By slight abuse of notation, we then denote by

$$Q_s=\sum_{e\in E_s}x_e^2$$

the diagonal bilinear form on $H_1(G,\Lambda)$ and note that it descends to the bilinear form on $H_1(\Gamma^1_s(X_0),\Lambda)$ from Definition 4.20 that we denote by the same symbol.

Proposition 4.30. *Let $s\in S$. The splitting ϕ_G of (4.8.1) is orthogonal with respect to Q_s. Moreover, the pullback $\phi_G^*Q_s$ agrees with $m\cdot d\cdot y_s^2$, where $y_s\in U^*_\Lambda$ is the linear form induced by the integral realization $S\to U^*$, $s\mapsto y_s$.*

Proof. Let $\alpha\in U_\Lambda$ and $\beta\in\ker(H_1(G,\Lambda)\to H_1(\Gamma(X_0),\Lambda))$. We will show that Q_s, viewed as a bilinear form, satisfies $Q_s(\phi_G(\alpha),\beta)=0$. Since Q_s is supported on edges of color s and the natural map $\mathrm{pr}_s\colon G\to\Gamma^1_s(X_0)$ is a morphism of S-colored graphs which is an isomorphism on edges of color s, we have

$$Q_s(\phi_G(\alpha),\beta)=Q_s(\mathrm{pr}_{s*}(\phi_G(\alpha)),\mathrm{pr}_{s*}(\beta)).$$

Since $\phi_s=\mathrm{pr}_s\circ\phi_G$ (see Proposition 4.29), we get $\mathrm{pr}_{s*}(\phi_G(\alpha))\in\phi_s(U_\Lambda)$; as

$$\mathrm{pr}_{s*}(\beta)\in\ker(H_1(\Gamma^1_s(X_0),\Lambda)\to H_1(\Gamma(X_0),\Lambda)=U_\Lambda),$$

the above right hand side vanishes by Lemma 4.26. Hence, the splitting ϕ_G is orthogonal with respect to Q_s.

Let now $\alpha,\beta\in U_\Lambda$. Then, by the same reasoning as above, we have

$$Q_s(\phi_G(\alpha),\phi_G(\beta))=Q_s(\mathrm{pr}_{s*}(\phi_G(\alpha)),\mathrm{pr}_{s*}(\phi_G(\beta)))=Q_s(\phi_s(\alpha),\phi_s(\beta)).$$

The above right hand side agrees with $m\cdot d\cdot y_s(\alpha)y_s(\beta)$, see Lemma 4.26. Hence, $\phi_G^*Q_s=m\cdot d\cdot y_s^2$, as we want. This concludes the proof of the proposition. □

Proof of Theorem 4.10. We replace the positive integer d by the positive multiple md. By Propositions 4.29 and 4.30, there is a decomposition

$$H_1(G, \Lambda) = U_\Lambda \oplus U'$$

which is orthogonal with respect to Q_s and such that Q_s restricts to $d \cdot y_s^2$ on U_Λ, for all $s \in S$. This says that there is a quadratic Λ-splitting of level d of $(\underline{R}, S)$ into the graph G, see Definition 4.6. This concludes the proof of the theorem. $\square$

5. From quadratic splittings to solutions in Albanese graphs

Throughout this section, ℓ denotes a prime number.

5.1. **Albanese graphs.**

Definition 5.1. Let S be a finite set and let G be an S-colored oriented graph. Let Λ be a ring. Then the *color profile map* (with coefficients in Λ) is the unique linear map

$$\lambda \colon C_1(G, \Lambda) \longrightarrow \Lambda^S$$

that sends an edge of color s, viewed as 1-chain via the given orientation, to the s-th basis vector e_s of Λ^S.

Assume in addition that G is Λ-weighted, i.e. for each edge e of G we are given a weight $a(e) \in \Lambda$. Then the *weighted color profile map* is the unique linear map

$$\lambda^w \colon C_1(G, \Lambda) \longrightarrow \Lambda^S$$

that sends an oriented edge e of color s to the element $a(e) \cdot e_s \in \Lambda^S$.

Remark 5.2. For a graph G with set of edges E, the choice of an orientation of G, i.e. an orientation for each edge, determines a canonical isomorphism $C_1(G, \Lambda) = \Lambda^E$. The color profile map λ can then explicitly be described as the composition

$$C_1(G, \Lambda) = \Lambda^E = \bigoplus_{s \in S} \Lambda^{E_s} \xrightarrow{\Sigma^S} \Lambda^S,$$

where Σ^S is induced by the sum maps $\Sigma_s \colon \Lambda^{E_s} \to \Lambda$, which maps a vector to the sum of its coordinates.

Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$, $s \mapsto y_s$. Dually, there is an inclusion $U \hookrightarrow \mathbb{Z}^S$, which identifies $U_\Lambda = U \otimes_{\mathbb{Z}} \Lambda$ with a subspace of Λ^S.

A crucial ingredient of the remaining sections is the following set of universal graphs associated to $(\underline{R}, S)$.

Definition 5.3. Let $\underline{R} = (\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$ and let $j \leq r$ be non-negative integers. The (ℓ^r, ℓ^j)*-Albanese graph* of $\underline{R}$ is the oriented S-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ whose set of vertices is given by

$$V = \mathbb{Z}^S/(\ell^j U + \ell^r \mathbb{Z}^S),$$

and such that two vertices $[v], [w] \in V$ are joined by an oriented edge of color s pointing from $[v]$ to $[w]$ if and only if $[w] = [v + e_s]$, where $e_s \in \mathbb{Z}^S$ denotes the s-th basis vector.

We will also write $\mathrm{Alb}_{\ell^r}(\underline{R}) := \mathrm{Alb}_{\ell^r,1}(\underline{R})$ and in particular $\mathrm{Alb}_{\ell}(\underline{R}) := \mathrm{Alb}_{\ell,1}(\underline{R})$.

Remark 5.4. The special case where $j = r = 0$ leads to the graph $\mathrm{Alb}_{1,1}(\underline{R})$, which has only one vertex and for each $s \in S$ a loop of color s. That is, $\mathrm{Alb}_{1,1}(\underline{R})$ is a wedge of $|S|$-many circles, one for each color $s \in S$.

Remark 5.5. For $\ell = 2$, the Albanese graph may have vertices $[v]$ and $[w]$ that are joined by multiple edges of the same color, but with reversed orientations. Such edges occur if $[v] = [v + 2e_s]$ and $[w] = [v + e_s] \neq [v]$.

Lemma 5.6. *The S-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ does not depend on the choice of integral realization $S \to U^*$. Moreover, its orientation is well-defined up to possibly reversing the orientation of all edges of some colors $s \in S' \subset S$ simultaneously.*

Proof. Fix a basis of $\underline{R}$. The chosen integral realization of $\underline{R}$ then corresponds to a matrix $(\mathbb{1}_g|D) \in \mathbb{Z}^{g\times|S|}$, where g denotes the rank of $\underline{R}$. By Lemma 2.2, another choice of integral realization of $\underline{R}$ leads (with respect to the same basis of $\underline{R}$) to a matrix $(\mathbb{1}_g|D')$ which may be obtained from $(\mathbb{1}_g|D)$ by multiplying some rows and columns by -1. Multiplying a row by -1 does not change the S-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ nor its orientation, while multiplying the s-th column of $(\mathbb{1}_g|D)$ by -1 does not change the S-colored graph, but reverses the orientation of all s-colored edges of the Albanese graph. The lemma follows. ☐

Assume now that Λ is a $\mathbb{Z}_{(\ell)}$-algebra with $\Lambda/\ell^r = \mathbb{Z}/\ell^r$. Then the set V of vertices of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ satisfies:

$$V = \mathbb{Z}^S/(\ell^j U + \ell^r \mathbb{Z}^S) = \Lambda^S/(\ell^j U_\Lambda + \ell^r \Lambda). \tag{5.1.1}$$

The natural commutative diagram

$$\begin{tikzcd} C_1(\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}), \Lambda) \arrow[r, "\lambda"] \arrow[d, "\partial"'] & \Lambda^S \arrow[d] \\ C_0(\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}), \Lambda) \arrow[r] & V = \Lambda^S/(\ell^j U_\Lambda + \ell^r \Lambda), \end{tikzcd}$$

in which the lower horizontal map sends $(a_v)_v \in \Lambda^V = C_0(\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}),\Lambda)$ to $\sum_v a_v \cdot v \in V$, shows that $\mathrm{im}(\lambda\colon H_1(\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}),\Lambda) \to \Lambda^S) \subset \ell^j U_\Lambda + \ell^r \Lambda^S$. The connected oriented S-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ is universal for this property:

Proposition 5.7 (Universal property)**.** *Let G be a connected, oriented, S-colored graph. Let Λ be a $\mathbb{Z}_{(\ell)}$-algebra with $\Lambda/\ell^r = \mathbb{Z}/\ell^r$. Assume that*

$$\mathrm{im}(\lambda\colon H_1(G,\Lambda) \to \Lambda^S) \subset \ell^j U_\Lambda + \ell^r \Lambda^S.$$

Then, for any $j \leq r$, the choice of a vertex v_0 of G defines a canonical map of oriented S-colored graphs

$$\mathrm{alb}\colon G \longrightarrow \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}),$$

which does not contract any edge. Explicitly, alb *maps a vertex v of G to the element $[\lambda(\gamma_v)] \in V$, where $\gamma_v \in C_1(G,\Lambda)$ is a 1-chain with $\partial\gamma_v = v - v_0$.*

Proof. Let us first show that the map alb is well-defined on the set of vertices. Since G is connected, we can find a 1-chain γ_v with $\partial\gamma_v = v - v_0$ for any vertex v of G. If γ_v' is another choice of such a 1-chain, then $\gamma_v - \gamma_v'$ is closed and so

$$\lambda(\gamma_v - \gamma_v') \in \ell^j U_\Lambda + \ell^r \Lambda^S$$

by assumptions. We deduce that

$$[\lambda(\gamma_v)] = [\lambda(\gamma_v')] \in V,$$

where we used (5.1.1). This proves that the map alb is well-defined on the set of vertices.

To prove that the map of vertices extends to a map of S-colored oriented graphs, let e be an edge of G of color s satisfying $\partial e = v_2 - v_1$ (with respect to the given orientation of e). Let further γ_{v_1} be a 1-chain with $\partial\gamma_{v_1} = v_1 - v_0$. Then $\gamma_{v_1} + e$ is a 1-chain with boundary $v_2 - v_0$. This shows $[\mathrm{alb}(v_1) + e_s] = [\mathrm{alb}(v_2)]$. Hence, $\mathrm{alb}(v_1)$ and $\mathrm{alb}(v_2)$ are joined by a unique oriented edge of color s which points from $\mathrm{alb}(v_1)$ to $\mathrm{alb}(v_2)$, and we may send e to this edge. Then alb respects the given orientations and S-colorings and hence defines a canonical map of oriented S-colored graphs, which does not contract any edge, as claimed. ∎

5.2. **ℓ^{j+1}-indivisible Λ-solutions in $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$.** Recall from Section 2.1.3, that a 1-chain $b \in C_1(G,\Lambda)$ of an S-colored graph G is of color $s \in S$, if it is a linear combination of oriented edges of color s. Note that this is strictly stronger than asking that the color profile of b is a multiple of e_s.

Definition 5.8. Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$. Let G be an S-colored oriented graph and let Λ be a $\mathbb{Z}_{(\ell)}$-algebra. A *Λ-solution of $(\underline{R}, S)$ in G* (or simply: a Λ-solution) is a collection of 1-chains $(b_s)_{s\in S}$, $b_s \in C_1(G,\Lambda)$, of G with coefficients in Λ such that the following hold:

(1) The 1-chain b_s has color s;
(2) Consider the inclusion $U_\Lambda \hookrightarrow \Lambda^S$ induced by the dual of the given realization. If $\sum_{s\in S} c_s e_s \in U_\Lambda$ for some $c_s \in \Lambda$, then $\sum_{s\in S} c_s b_s \in H_1(G,\Lambda)$ is closed.

For $i \geq 0$, we say that a Λ-solution $(b_s)_{s\in S}$ is

- *ℓ^i-indivisible* if, for all $s \in S$, the color profile of b_s is not ℓ^i-divisible: $\ell^i \nmid \lambda(b_s)$,
- *constant modulo ℓ^i* if $\lambda(b_s) \equiv \lambda(b_t) \bmod \ell^i$ for all $s, t \in S$.

Remark 5.9. Item (2) can be reformulated as follows. Let $E = \bigsqcup_{s\in S} E_s$ be the set of edges of G and consider the linear map $\Psi : \Lambda^S \to \Lambda^E = C_1(G,\Lambda)$, which sends the basis vector e_s to the 1-chain b_s. Then the restriction of Ψ to U_Λ lands in $H_1(G,\Lambda)$ and so there is a unique map $U_\Lambda \to H_1(G,\Lambda)$ which makes the following diagram commutative:

$$\begin{tikzcd} U_\Lambda \arrow[r, hook] \arrow[d, "\exists!"] & \Lambda^S \arrow[d, "\Psi"] \\ H_1(G,\Lambda) \arrow[r, hook] & \Lambda^E. \end{tikzcd}$$

Using this description, it easily follows from Lemma 2.2 that the existence of an ℓ^i-indivisible Λ solution of $(\underline{R}, S)$ in a graph G does not depend on the chosen integral realization. Indeed, the choice of a basis of $\underline{R}$ yields an isomorphism $U_\Lambda \cong \Lambda^g$, where g denotes the rank of $\underline{R}$, and passing to another integral realization changes the matrix that represents the top horizontal map in the given basis only by multiplying some rows and columns by -1.

The main result of this section is the following theorem.

Theorem 5.10. *Let $(\underline{R}, S)$ be a regular loopless matroid with integral realization $S \to U^*$. Let ℓ be a prime, r a positive integer and Λ be a $\mathbb{Z}_{(\ell)}$-algebra with $\Lambda \subset \mathbb{R}$ and $\Lambda/\ell^r = \mathbb{Z}/\ell^r$. Assume that there is a positive integer d and a graph G, such that $(\underline{R}, S)$ admits a quadratic Λ-splitting of level d in G (see Definition 4.6). Write $d = \ell^j d'$ for an integer d' coprime to ℓ. Then, for all $j+1 \leq r$, the (ℓ^r, ℓ^j)-Albanese graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ admits an ℓ^{j+1}-indivisible Λ-solution of $(\underline{R}, S)$ which is constant modulo ℓ^r (see Definitions 5.3 and 5.8).*

The rest of this section is devoted to a proof of the above theorem; to this end, Λ will, for the remainder of this section, denote a $\mathbb{Z}_{(\ell)}$-algebra with $\Lambda \subset \mathbb{R}$ and $\Lambda/\ell^r = \mathbb{Z}/\ell^r$. (There is no harm in assuming $\Lambda = \mathbb{Z}_{(\ell)}$, which suffices for our applications, but we prefer to make the precise assumptions, needed in our argument, transparent.) Recall also that the data of a quadratic Λ-splitting of $\underline{R}$ in G comes in particular with an S-coloring of G and so we may from now on view G as an S-colored graph.

As a first reduction step, we reduce to the case where G is connected. Indeed, if it is not, then we pick a vertex in each connected component and glue these vertices to get a connected S-colored graph G'. We then replace G by G'. This does not change the cohomology of G and hence does not change the fact that $(\underline{R}, S)$ admits a quadratic Λ-splitting of level d in G. So we henceforth assume that G is connected.

5.3. **Characteristic 1-chain of color s.** Denote the set of edges of G by E and the set of edges of color s by E_s. Since G is S-colored, $E = \sqcup_{s\in S} E_s$. We further fix once and for all an orientation on the set of edges of G. This induces a canonical embedding

$$H_1(G, \Lambda) \subset \Lambda^E = C_1(G, \Lambda).$$

Proposition 5.11. *Consider an edge $e \in E_s$ of color s of G and the corresponding coordinate function $x_e\colon \Lambda^E \to \Lambda$. Then the linear form $x_e|_{U_\Lambda} \in U_\Lambda^*$, given as the composition*

$$U_\Lambda \hookrightarrow H_1(G, \Lambda) \subset \Lambda^E \longrightarrow \Lambda, \quad u \mapsto x_e(u),$$

is a multiple of y_s, where y_s are the linear forms of the integral realization $S \to U^*$, $s \mapsto y_s$, of $(\underline{R}, S)$.

Proof. We can extend the element $s \in S$ to a basis $S' \subset S$ of the matroid $(\underline{R}, S)$. Assume without loss of generality that $s = 1$ and $S' = \{1, \dots, g\}$. Then the linear forms $y_1, \dots, y_g$ form a basis of U^*. It follows that there are unique elements $a_{ei} \in \Lambda$ with

$$x_e|_{U_\Lambda} = \sum_{i=1}^{g} a_{ei} y_i.$$

Since $U_\Lambda \hookrightarrow H_1(G, \Lambda)$ forms part of the data of a quadratic splitting of $(\underline{R}, S)$ in G of level d, we have

$$d \cdot y_1^2 = \sum_{e\in E_s} x_e^2|_U,$$

see Definition 4.6 (and recall that $s = 1$). We thus get

$$d \cdot y_1^2 = \sum_{e\in E_s} x_e^2|_U = \sum_{e\in E_s} \left(\sum_{i=1}^{g} a_{ei} y_i \right)^2.$$

The above equality is an equality of quadratic forms on U_Λ; since Λ has characteristic zero and $y_1, \dots, y_g$ is a basis of U_Λ^*, we see that the equality holds in fact in the polynomial ring $\Lambda[y_1, \dots, y_g]$. Since $\Lambda \subset \mathbb{R}$, we then deduce from the elementary Lemma 5.12 below that $a_{ei} = 0$ for all $i \geq 2$ and all $e \in E_s$. This proves that $x_e|_U$ is a multiple of y_s. $\square$

Lemma 5.12. *Let Λ be a subring of $\mathbb{R}$ and consider the polynomial ring $\Lambda[y_1,\dots,y_g]$ in g variables. Let $(a_{ji})\in\Lambda^{m\times g}$ be an $m\times g$ matrix and assume there is some $d\in\Lambda$ with*

$$d\cdot y_1^2=\sum_{j=1}^m\left(\sum_{i=1}^g a_{ji}y_i\right)^2.$$

Then $a_{ji}=0$ for all $i\geq 2$ and $j\geq 1$.

Proof. We have

$$\sum_{j=1}^m\left(\sum_{i=1}^g a_{ji}y_i\right)^2=\sum_{i=1}^g\left(\sum_{j=1}^m a_{ji}^2\right)y_i^2+\sum_{i_1<i_2}c_{i_1,i_2}y_{i_1}y_{i_2}$$

for some $c_{i_1,i_2}\in\Lambda$. Since the above expression agrees with $d\cdot y_1^2$ by assumption, we find that the mixed terms $c_{i_1,i_2}y_{i_1}y_{i_2}$, $i_1<i_2$, vanish and, in addition, $\sum_{j=1}^m a_{ji}^2=0$ for all $i\geq 2$. Since $\Lambda\subset\mathbb{R}$, this is only possible if $a_{ji}=0$ for all $i\geq 2$, as claimed. □

By Proposition 5.11, for each $s\in S$ and each edge $i\in E_s$ of G of color s, there is a unique element $a(i)\in\Lambda$ such that

$$x_i|_{U_\Lambda}=a(i)y_s\in U_\Lambda^*. \tag{5.3.1}$$

The coefficients $a(i)$ play an important role and lead in particular to the following notion:

Definition 5.13. The *characteristic* 1-chain of color s is the 1-chain

$$b_s:=\sum_{i\in E_s}a(i)\cdot i\in C_1(G,\Lambda).$$

Associated to the characteristic 1-chains b_s of color s, there is the linear map

$$\Psi:\Lambda^S\longrightarrow\Lambda^E=\bigoplus_{s\in S}\Lambda^{E_s},\qquad\sum_{s\in S}c_se_s\mapsto\sum_{s\in S}c_sb_s,$$

which identifies to the direct sum of the maps $\Lambda\to\Lambda^{E_s}$, $1\mapsto b_s$. We may then consider the following diagram

$$\begin{tikzcd} U_\Lambda \arrow[r, hook] \arrow[d, hook] & \Lambda^S \arrow[d, "\Psi"] \\ H_1(G,\Lambda) \arrow[r, hook] & \Lambda^E=\bigoplus_{s\in S}\Lambda^{E_s} \end{tikzcd} \tag{5.3.2}$$

where the top horizontal arrow is induced by the dual of the realization $S\to U^*$ of $\underline{R}$, the lower horizontal arrow is induced by the choice of orientation of G, and the left vertical map is induced by the fact that there is a Λ-splitting of $(\underline{R},S)$ in G of level d.

Proposition 5.14. *The diagram (5.3.2) commutes. In other words, if $u=\sum_s c_se_s\in U_\Lambda\subset\Lambda^S$, then the 1-chain $\sum_s c_sb_s\in C_1(G,\Lambda)=\Lambda^E$ is closed and agrees in fact with $u\in U_\Lambda$ via the embedding $U_\Lambda\subset H_1(G,\Lambda)$.*

Proof. Let $u \in U_\Lambda$. We first think about U_Λ as a subspace of Λ^S and write

$$u = \sum_s c_s e_s \in U_\Lambda \subset \Lambda^S$$

for some $c_s \in \Lambda$. This means that $y_s(u) = c_s$ for all $s \in S$.

Now view U_Λ as a subspace of $H_1(G, \Lambda)$. By (5.3.1), we have for all $i \in E_s$ the identity

$$x_i|_{U_\Lambda} = a(i) y_s \in U_\Lambda^*.$$

Hence, for all $s \in S$ and $i \in E_s$, we find

$$x_i(u) = a(i) \cdot y_s(u) = a(i) \cdot c_s \in \Lambda.$$

This in turn precisely means that

$$u = \sum_{s \in S} \sum_{i \in E_s} c_s \cdot a(i) \cdot i = \sum_{s \in S} c_s \cdot b_s \in U_\Lambda.$$

This proves the proposition. □

5.4. **Weighted color profile.** In this section, we use the elements $a(i) \in \Lambda$ from (5.3.1) to define a Λ-weighting on the oriented S-colored graph G as follows.

Definition 5.15. We define the *weight* of an edge $i \in E_s$ to be the element $a(i) \in \Lambda$, where $a(i)$ is as in (5.3.1).

Associated to this weighting of G, there is a weighted color profile map

$$\lambda^w \colon C_1(G, \Lambda) \longrightarrow \Lambda^S, \tag{5.4.1}$$

which is the unique Λ-linear map which sends an edge $i \in E_s$ of color s to $a(i) \cdot e_s$ (see also Definition 5.1). The s-th component of this map is denoted by

$$\lambda^w_s \colon C_1(G, \Lambda) \longrightarrow \Lambda.$$

We then have $\lambda^w(x) = \sum_s \lambda^w_s(x) e_s$ for $x \in C_1(G, \Lambda)$.

Recall, moreover, the Q_s-orthogonal decomposition

$$H_1(G, \Lambda) = U_\Lambda \oplus U',$$

induced by the quadratic Λ-splitting of $(\underline{R}, S)$ in G, see Definition 4.6.

Proposition 5.16. *The weighted color profile map λ^w from (5.4.1) has the following properties:*

1. *If $u' \in U'$, then $\lambda^w(u') = 0$.*
2. *The characteristic 1-cycle b_s of color s (see Definition 5.13) satisfies $\lambda^w_s(b_s) = d$.*
3. *The image of the restriction of λ^w to $H_1(G, \Lambda)$ agrees with $d \cdot U_\Lambda \subset \Lambda^S$.*

Proof. By assumption in Theorem 5.10, $(\underline{R}, S)$ is loopless. By Lemma 2.3, this means that for each $s \in S$, there exists some $u_s \in U_\Lambda$ and elements $c_t \in \Lambda$, $t \in S \setminus s$, with

$$u_s = e_s + \sum_{t \in S \setminus s} c_t e_t \in U_\Lambda. \tag{5.4.2}$$

If we view u_s as an element of $U_\Lambda \subset H_1(G, \Lambda)$, then this implies by Proposition 5.14 that the s-colored part of the closed 1-chain u_s agrees with the characteristic 1-chain b_s, see Definition 5.13.

Consider the bilinear form $Q_s = \sum_{i \in E_s} x_i^2$ on $C_1(G, \Lambda)$. We then have

$$Q_s(u_s, \alpha) = Q_s(b_s, \alpha) \qquad \text{for all } \alpha \in C_1(G, \Lambda),$$

because Q_s is supported on the edges of color s. Moreover, if we write $\alpha = \sum_{i \in E} \alpha_i \cdot i$ as a linear combination of edges, then, by linearity, and because $Q_s = \sum_{i \in E_s} x_i^2$, we get

$$Q_s(b_s, \alpha) = \sum_{i \in E_s} \alpha_i Q_s(b_s, i) = \sum_{i \in E_s} \alpha_i Q_s(a(i)i, i) = \sum_{i \in E_s} a(i) \cdot \alpha_i.$$

The above right hand side agrees with $\lambda_s^w(\alpha)$. We have thus shown that

$$Q_s(u_s, \alpha) = \lambda_s^w(\alpha) \qquad \text{for all } \alpha \in C_1(G, \Lambda). \tag{5.4.3}$$

Since $u_s \in U_\Lambda$ and the decomposition $H_1(G, \Lambda) = U_\Lambda \oplus U'$ is Q_s-orthogonal, (5.4.3) implies that

$$\lambda^w(u') = Q_s(u_s, u') = 0 \qquad \text{for all } u' \in U'.$$

This proves item (1) in the proposition.

To prove item (2), note that $\lambda_s^w(b_s) = \lambda_s^w(u_s)$, because the s-colored part of the closed 1-chain u_s is given by b_s. Moreover, $\lambda_s^w(u_s) = Q_s(u_s, u_s)$, by (5.4.3). Finally, $Q_s(u_s, u_s) = d \cdot y_s(u_s)^2$, because $Q_s|_{U_\Lambda} = d \cdot y_s^2$. By (5.4.2), $y_s(u_s) = 1$ and so

$$\lambda_s^w(b_s) = d \cdot y_s(u_s)^2 = d.$$

This proves item (2).

Finally, to prove item (3), recall the commutative diagram (5.3.2), see Proposition 5.14. By item (1), proven above, and the decomposition $H_1(G, \Lambda) = U_\Lambda \oplus U'$, the image of λ^w agrees with the image of its restriction to U_Λ. Let now $u \in U_\Lambda$, view U_Λ as a subspace of Λ^S, and write $u = \sum c'_s e_s$ for some $c'_s \in \Lambda$. By the commutative diagram (5.3.2), we then have

$$\lambda^w(u) = \lambda^w\left(\sum_{s \in S} c'_s b_s\right) = \sum_{s \in S} c'_s \lambda_s^w(b_s) \cdot e_s.$$

By item (2), proven above, $\lambda_s^w(b_s) = d$ and we get

$$\lambda^w(u) = \sum_{s \in S} c'_s d \cdot e_s = d \sum_{s \in S} c'_s e_s = d \cdot u \in \Lambda^S.$$

This proves item (3) and hence concludes the proof of the proposition. $\square$

5.5. **Refinement of G and its Albanese image.** In this section we finish the proof of Theorem 5.10. To this end, we write $d = \ell^j d'$ for an integer d' coprime to ℓ and we fix an integer $r \geq j+1$. Our goal is to show that the (ℓ^r, ℓ^j)-Albanese graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ admits an ℓ^{j+1}-indivisible Λ-solution (see Definitions 5.3 and 5.8).

To begin with, for each edge i of G, we let $\hat{a}(i) \in \mathbb{Z}$ denote the smallest positive integer with the property that

$$\hat{a}(i) \equiv a(i) \bmod \ell^r,$$

where $a(i) \in \Lambda$ are the elements from (5.3.1). Note that $\hat{a}(i)$ exists because $\Lambda/\ell^r = \mathbb{Z}/\ell^r$ by assumption.

Definition 5.17. We denote by $\hat{G}$ the oriented S-colored graph that is given by the refinement of G, where each edge $i \in E_s$ of color s of G is replaced by a chain of $\hat{a}(i)$-many consecutive edges of the same color and same orientation.

We define

$$\Xi : C_1(G, \Lambda) \longrightarrow C_1(\hat{G}, \Lambda)$$

as the unique linear map which sends an edge $i \in E_s$ of G of color s to the chain of $\hat{a}(i)$-many edges of $\hat{G}$ of the same color and same orientation that corresponds to i in the refinement $\hat{G}$ of G. Note that Ξ induces the canonical isomorphism

$$\Xi|_{H_1(G,\Lambda)} \colon H_1(G, \Lambda) \xrightarrow{\cong} H_1(\hat{G}, \Lambda)$$

induced by the homeomorphism $G \xrightarrow{\approx} \hat{G}$ given by the fact that $\hat{G}$ is a refinement of G.

Note that $\hat{G}$ is, by definition, unweighted. In fact, the refinement is chosen in such a way that the weighted color profile of G compares to the (unweighted) color profile map

$$\lambda : C_1(\hat{G}, \Lambda) \longrightarrow \Lambda^S$$

of $\hat{G}$, as follows:

Lemma 5.18. *The following diagram commutes modulo ℓ^r:*

$$\begin{tikzcd} C_1(G,\Lambda) \arrow[d, "\Xi"'] \arrow[dr, "\lambda^w"] & \\ C_1(\hat{G},\Lambda) \arrow[r, "\lambda"'] & \Lambda^S. \end{tikzcd}$$

Proof. By linearity, it suffices to check that $(\lambda \circ \Xi)(i) \equiv \lambda^w(i) \bmod \ell^r$ for a single edge $i \in E_s$ of color s of G. Then $\Xi(i)$ is a chain of $\hat{a}(i)$-many edges of the same color and same orientation. Hence, $\lambda(\Xi(i)) = \hat{a}(i) \cdot e_s$. On the other hand, $\lambda^w(i) = a(i) \cdot e_s$. The result thus follows from the fact that $\hat{a}(i) \equiv a(i) \bmod \ell^r$. $\square$

Proof of Theorem 5.10. By Lemma 5.18, item (3) in Proposition 5.16 implies that

$$\mathrm{im}(\lambda\colon H_1(\hat{G},\Lambda) \longrightarrow \Lambda^S) \equiv \ell^j U_\Lambda \mod \ell^r \Lambda^S,$$

where we used that $d = \ell^j d'$, where d' is coprime to ℓ and hence invertible in Λ. Thus,

$$\mathrm{im}(\lambda\colon H_1(\hat{G},\Lambda) \longrightarrow \Lambda^S) \subset \ell^j U_\Lambda + \ell^r \Lambda^S. \tag{5.5.1}$$

Recall that G is connected by the first reduction step in the proof of Theorem 5.10 (see the paragraph after the theorem). Hence, $\hat{G}$ is connected as well. By Proposition 5.7, (5.5.1) implies that the choice of a vertex of $\hat{G}$ defines a canonical morphism of oriented S-colored graphs

$$\mathrm{alb}\colon \hat{G} \longrightarrow \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}).$$

Recall the characteristic 1-chain $b_s \in C_1(G,\Lambda)$ from Definition 5.13 and define

$$\hat{b}_s := \Xi(b_s) \in C_1(\hat{G},\lambda).$$

(By the definition of Ξ, $\hat{b}_s$ is the 1-chain on $\hat{G}$ that is obtained from b_s by replacing each edge $i \in E_s$ of G by the corresponding chain of $\hat{a}(i)$-many edges of $\hat{G}$ of the same color and same orientation.)

Let now $c_s \in \Lambda$, for $s \in S$, be such that $\sum_{s\in S} c_s e_s \in U_\Lambda \subset \Lambda^S$. By Proposition 5.14, $\sum_{s\in S} c_s b_s \in C_1(G,\Lambda)$ is closed. Since Ξ respects closedness, we find that

$$\Xi(\textstyle\sum_{s\in S} c_s b_s) = \sum_{s\in S} c_s \hat{b}_s \in H_1(\hat{G},\Lambda).$$

This shows that $(\hat{b}_s)_{s\in S}$ is a Λ-solution of $(\underline{R},S)$ in $\hat{G}$, see Definition 5.8. By Lemma 5.18, we have $\lambda(\hat{b}_s) \equiv \lambda^w(b_s) \bmod \ell^r$. By item (2) in Proposition 5.16, we deduce that

$$\lambda(\hat{b}_s) \equiv d e_s \bmod \ell^r. \tag{5.5.2}$$

Since $d = \ell^j d'$ with d' coprime to ℓ, and since $r \geq j+1$ by assumption, we see that the Λ-solution $(\hat{b}_s)_{s\in S}$ of $(\underline{R},S)$ in $\hat{G}$ is in fact ℓ^{j+1}-indivisible.

Because the Albanese morphism alb is a morphism of oriented S-colored graphs, the collection of 1-chains

$$\mathrm{alb}_* \hat{b}_s \in C_1(\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}),\Lambda)$$

with $s \in S$ is then a Λ-solution of $(\underline{R},S)$ in $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ which is still ℓ^{j+1}-indivisible, because alb does not contract any edge. For the same reason, (5.5.2) implies that the solution is constant modulo ℓ^r in the sense of Definition 5.8. This concludes the proof of Theorem 5.10. □

6. From ℓ^j-indivisible solutions to ℓ-indivisible solutions

The main result of this section is the following ℓ-power descent theorem for solutions in Albanese graphs, improving Theorem 5.10 from the previous section.

Theorem 6.1. *Let $(\underline{R}, S)$ be a regular loopless matroid with integral realization $S \to U^*$. Let $j \le i \le r$ be non-negative integers, ℓ a prime and let Λ be an $\mathbb{Z}_{(\ell)}$-algebra. Assume that $(\underline{R}, S)$ has an ℓ^i-indivisible Λ-solution in $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ which is constant modulo ℓ^r. Then $(\underline{R}, S)$ has an ℓ^{i-j}-indivisible Λ-solution in $\mathrm{Alb}_{\ell^{r-j},1}(\underline{R})$ which is constant modulo ℓ^{r-j}.*

6.1. Albanese tori.

Definition 6.2. Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$, and let $j \le r$ be non-negative integers. Then the (ℓ^r, ℓ^j)-*Albanese torus* of $(\underline{R}, S)$ is the real $|S|$-dimensional torus

$$\mathcal{T}_{\ell^r,\ell^j}(\underline{R}) := \mathbb{R}^S/(\ell^j U + \ell^r \mathbb{Z}^S).$$

Remark 6.3. It follows from Lemma 2.2 that $\mathcal{T}_{\ell^r,\ell^j}(\underline{R})$ depends only on $(\underline{R}, S)$ and not on the chosen integral realization $S \to U^*$, cf. proof of Lemma 5.6.

Remark 6.4. It will frequently be convenient to endow $\mathcal{T}_{\ell^r,\ell^j}(\underline{R})$ with the structure of a polyhedral complex, induced by the unit cube tiling of $\mathbb{R}^S$. For instance, using this polyhedral structure, the Albanese graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ from Definition 5.3 is easily identified to the 1-skeleton of $\mathcal{T}_{\ell^r,\ell^j}(\underline{R})$.

The remainder of Section 6 is devoted to a proof of Theorem 6.1. To this end we fix a regular loopless matroid $(\underline{R}, S)$ with integral realization $S \to U^*$. We then write for simplicity

$$\mathrm{Alb}_{\ell^r,\ell^j} := \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}) \quad \text{and} \quad \mathcal{T}_{\ell^r,\ell^j} := \mathcal{T}_{\ell^r,\ell^j}(\underline{R}).$$

By Remark 6.4, there is a natural inclusion of polyhedral complexes

$$\mathrm{Alb}_{\ell^r,\ell^j} \hookrightarrow \mathcal{T}_{\ell^r,\ell^j}.$$

We consider the canonical topological covering map

$$\xi : \mathcal{T}_{\ell^r,\ell^j} = \mathbb{R}^S/(\ell^j U + \ell^r \mathbb{Z}^S) \longrightarrow \mathcal{T}_{\ell^j,\ell^j} = \mathbb{R}^S/\ell^j \mathbb{Z}^S,$$

given by quotiening out $\ell^j \mathbb{Z}^S/\ell^j U + \ell^r \mathbb{Z}^r$. This map induces topological coverings of the respective Albanese graphs, that we denote by the same symbol:

$$\xi \colon \mathrm{Alb}_{\ell^r,\ell^j} \longrightarrow \mathrm{Alb}_{\ell^j,\ell^j}.$$

6.2. **A refinement of the Albanese graph and the Albanese torus.** From now on we fix the non-negative integers $j \leq r$ from Theorem 6.1.

Definition 6.5. For non-negative integers $a \leq b$, let $\widehat{\mathrm{Alb}}_{\ell^b,\ell^a}$ be the S-colored graph obtained from $\mathrm{Alb}_{\ell^b,\ell^a}$ by replacing each edge by a chain of ℓ^j consecutive edges of the same color and the same orientation.

Similarly, we denote by $\widehat{\mathcal{T}}_{\ell^b,\ell^a}$ the polyhedral structure on $\mathbb{R}^S/(\ell^a U + \ell^b \mathbb{Z}^S)$, induced by the tiling of $\mathbb{R}^S$ with cubes of side length $1/\ell^j$.

We view $\mathcal{T}_{\ell^b,\ell^a} = \mathbb{R}^S/(\ell^a U + \ell^b \mathbb{Z}^S)$ as the polyhedral complex, induced by the unit cube tiling on $\mathbb{R}^S$, cf. Remark 6.4.

Lemma 6.6. *Let $a \leq b$ be non-negative integers. Then multiplication by ℓ^j induces a canonical identification of polyhedral complexes*

$$\widehat{\mathcal{T}}_{\ell^b,\ell^a} \xrightarrow{\cong} \mathcal{T}_{\ell^{b+j},\ell^{a+j}}.$$

Precomposing this map with the inclusion of $\widehat{\mathrm{Alb}}_{\ell^b,\ell^a}$ into the 1-skeleton of $\widehat{\mathcal{T}}_{\ell^b,\ell^a}$ induces a natural embedding of oriented S-colored graphs

$$\iota\colon \widehat{\mathrm{Alb}}_{\ell^b,\ell^a} \hookrightarrow \mathrm{Alb}_{\ell^{b+j},\ell^{a+j}}$$

Proof. The first claim is clear. The second one follows because the 1-skeleton of $\mathcal{T}_{\ell^{b+j},\ell^{a+j}}$ is canonically isomorphic to $\mathrm{Alb}_{\ell^{b+j},\ell^{a+j}}$, see Remark 6.4. □

Lemma 6.7. *Assume that $(\underline{R}, S)$ has an ℓ^i-indivisible Λ-solution in $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$ which is constant modulo ℓ^r. Then $(\underline{R}, S)$ has an ℓ^{i-j}-indivisible Λ-solution in $\mathrm{Alb}_{\ell^{r-j},1}$ which is constant modulo ℓ^{r-j}.*

Proof. Recall that $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$ is obtained from $\mathrm{Alb}_{\ell^{r-j},1}$ by dividing each edge e of $\mathrm{Alb}_{\ell^{r-j},1}$ into a chain of ℓ^j consecutive edges of the same orientation and the same color. Since $(\underline{R}, S)$ is loopless, the natural linear map $C_1(\mathrm{Alb}_{\ell^{r-j},1}, \Lambda) \to C_1(\widehat{\mathrm{Alb}}_{\ell^{r-j},1}, \Lambda)$, that sends an edge to the sum of the consecutive edges that divide it, induces a bijection between the spaces of Λ-solutions. The lemma follows from this. □

In order to prove Theorem 6.1, our goal is to produce solutions of $\mathrm{Alb}_{\ell^r,\ell^j}$ that are supported on the image of $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$ and to apply Lemma 6.7 to conclude. The idea is to push solutions from $\mathrm{Alb}_{\ell^r,\ell^j}$ to $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$, viewed as a subgraph of $\mathrm{Alb}_{\ell^r,\ell^j}$ via Lemma 6.6, along a suitable map of the ambient Albanese torus, which we discuss next.

6.3. **Homotopy.** For a real number x, we denote by $\lceil x \rceil$ the round-up of x. For a vector $x \in \mathbb{R}^S$ we denote by $\lceil x \rceil$ the vector, given by applying $\lceil - \rceil$ componentwise. We then consider the fundamental domain $[0, \ell^j]^S$ of $\mathcal{T}_{\ell^j,\ell^j} = \mathbb{R}^S/\ell^j\mathbb{Z}^S$ together with the continuous self-map

$$\tilde{h}\colon [0, \ell^j]^S \longrightarrow [0, \ell^j]^S, \tag{6.3.1}$$

given by

$$\tilde{h}(x) := \begin{cases} \ell^j \cdot x & \text{if } x \in [0,1]^S \\ \ell^j \lceil x/\ell^j \rceil & \text{otherwise.} \end{cases}$$

Lemma 6.8. *The map $\tilde{h}$ descends to a continuous self-map*

$$h\colon \mathcal{T}_{\ell^j,\ell^j} = \mathbb{R}^S/\ell^j\mathbb{Z}^S \longrightarrow \mathcal{T}_{\ell^j,\ell^j} = \mathbb{R}^S/\ell^j\mathbb{Z}^S$$

which is homotopic to the identity.

Proof. Since $\tilde{h}$ is defined componentwise, the lemma reduces to the claim that the self-map of the interval $[0, \ell^j]$, which maps $x \in [0,1]$ to $\ell^j x$ and maps $x \in [1, \ell^j]$ to $\ell^j \lceil x/\ell^j \rceil = \ell^j$, is continuous and descends to a self-map of the circle $\mathbb{R}/\ell^j\mathbb{Z}$ which is homotopic to the identity. This claim is clear and so the lemma follows. □

Remark 6.9. We can describe the map h as follows: The map $\tilde{h}$ is induced by first retracting the cube $[0, \ell^j]^S$ of side length ℓ^j suitably to the unit cube $[0,1]^S$ and then expanding the unit cube to $[0, \ell^j]^S$ by multiplication by ℓ^j. See Figure 3.

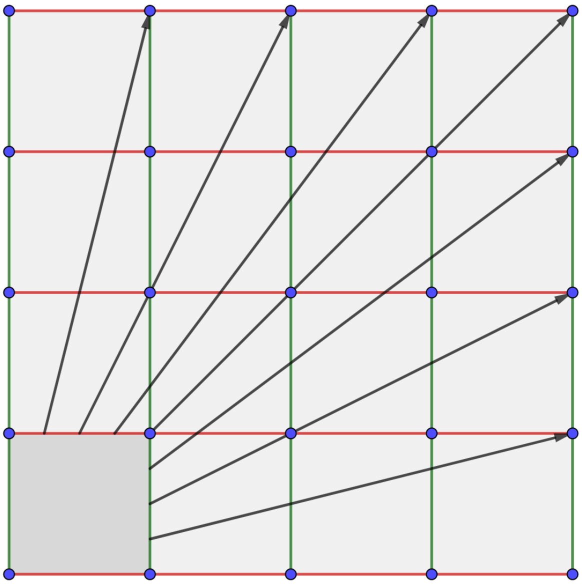

FIGURE 3. Depiction of the map $h\colon [0, 2^2]^2 \to [0, 2^2]^2$ (i.e., $\ell = j = |S| = 2$).

By Lemma 6.8, the map $\tilde{h}$ from (6.3.1) extends to a continuous map

$$\tilde{h}\colon \mathbb{R}^S \longrightarrow \mathbb{R}^S, \tag{6.3.2}$$

which is equivariant with respect to the natural $\ell^j\mathbb{Z}^S$-action on source and target that is given by translation; we denote this map by slight abuse of notation by the same letter.

Recall the graph $\mathrm{Alb}_{1,1}$, which is a graph with one vertex and one oriented edge for each $s \in S$; i.e. it is a wedge of $|S|$-many circles. We consider the refinement $\widehat{\mathrm{Alb}}_{1,1}$ of $\mathrm{Alb}_{1,1}$ from Definition 6.5. By Lemma 6.6, there is a canonical inclusion of oriented S-colored graphs

$$\widehat{\mathrm{Alb}}_{1,1} \lhook\joinrel\longrightarrow \mathrm{Alb}_{\ell^j,\ell^j} \tag{6.3.3}$$

which we use to identify $\widehat{\mathrm{Alb}}_{1,1}$ with a subgraph of $\mathrm{Alb}_{\ell^j,\ell^j}$. Via the inclusion $\mathrm{Alb}_{\ell^j,\ell^j} \hookrightarrow \mathcal{T}_{\ell^j,\ell^j}$, we thus get a canonical embedding of topological spaces

$$\widehat{\mathrm{Alb}}_{1,1} \lhook\joinrel\longrightarrow \mathcal{T}_{\ell^j,\ell^j}. \tag{6.3.4}$$

As before, we identify the refinement $\widehat{\mathrm{Alb}}_{\ell^j,\ell^j}$ of $\mathrm{Alb}_{\ell,\ell^j}$ canonically with a subgraph of the 1-skeleton of $\widehat{\mathcal{T}}_{\ell^j,\ell^j}$. This yields a canonical embedding of topological spaces:

$$\widehat{\mathrm{Alb}}_{\ell^j,\ell^j} \lhook\joinrel\longrightarrow \widehat{\mathcal{T}}_{\ell^j,\ell^j} \approx \mathcal{T}_{\ell^j,\ell^j}. \tag{6.3.5}$$

Lemma 6.10. *The continuous map $h\colon \mathcal{T}_{\ell^j,\ell^j} \to \mathcal{T}_{\ell^j,\ell^j}$ restricts to a map of oriented S-colored graphs*

$$h\colon \widehat{\mathrm{Alb}}_{\ell^j,\ell^j} \longrightarrow \widehat{\mathrm{Alb}}_{1,1},$$

where both sides are viewed as subspaces of $\mathcal{T}_{\ell^j,\ell^j}$ via the inclusions (6.3.5) and (6.3.4), respectively.

Proof. This follows easily from the construction of $\tilde{h}$ and h. $\square$

Lemma 6.11. *The following diagram is Cartesian*

$$\begin{tikzcd}
\widehat{\mathrm{Alb}}_{\ell^{r-j},1} \arrow[r, hook, "\iota"] \arrow[d] & \mathrm{Alb}_{\ell^r,\ell^j} \arrow[d, "\xi"] \\
\widehat{\mathrm{Alb}}_{1,1} \arrow[r, hook, "\iota"] & \mathrm{Alb}_{\ell^j,\ell^j}
\end{tikzcd}$$

where the horizontal maps are the embeddings from Lemma 6.6 and the vertical maps are the natural covering maps, respectively.

Proof. By Lemma 6.6 we have a canonical identification of polyhedral complexes $\widehat{\mathcal{T}}_{\ell^{r-j},1} \cong \mathcal{T}_{\ell^r,\ell^j}$ and $\widehat{\mathcal{T}}_{1,1} \cong \mathcal{T}_{\ell^j,\ell^j}$. Using this, we get a natural commutative diagram

$$\begin{tikzcd} \widehat{\mathrm{Alb}}_{\ell^{r-j},1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{\mathcal{T}}_{\ell^{r-j},1} \cong \mathcal{T}_{\ell^r,\ell^j} \arrow[d, "\xi"] \arrow[dr, phantom, "\square"] & \mathrm{Alb}_{\ell^r,\ell^j} \arrow[l, hook'] \arrow[d, "\xi"] \\ \widehat{\mathrm{Alb}}_{1,1} \arrow[r, hook] & \widehat{\mathcal{T}}_{1,1} \cong \mathcal{T}_{\ell^j,\ell^j} & \mathrm{Alb}_{\ell^j,\ell^j} \arrow[l, hook'] \end{tikzcd}$$

where the horizontal maps are the natural embeddings (which are maps of polyhedral complexes) and the vertical maps are induced by the covering map ξ. Note moreover that the squares on the left and the right are Cartesian. The claim in the lemma then follows because the image of the left lower horizontal map is contained in the image of the right lower horizontal one. $\square$

Definition 6.12. We define $H\colon \mathcal{T}_{\ell^r,\ell^j} \to \mathcal{T}_{\ell^r,\ell^j}$ to be the map of topological spaces that is induced by the equivariant map $\tilde{h}\colon \mathbb{R}^S \to \mathbb{R}^S$ from (6.3.2).

Lemma 6.13. *The map H is homotopic to the identity and makes following diagram commutative:*

$$\begin{tikzcd} \mathcal{T}_{\ell^r,\ell^j} \arrow[r, "H"] \arrow[d, "\xi"] & \mathcal{T}_{\ell^r,\ell^j} \arrow[d, "\xi"] \\ \mathcal{T}_{\ell^j,\ell^j} \arrow[r, "h"] & \mathcal{T}_{\ell^j,\ell^j}. \end{tikzcd}$$

Proof. Commutativity of the diagram is clear, because h is induced by $\tilde{h}$ as well. The fact that H is homotopic to the identity follows from the homotopy lifting property together with the fact that h is homotopic to the identity, see Lemma 6.8. $\square$

It follows from Lemmas 6.10, 6.11 and 6.13 that H restricts to a morphism of oriented S-colored graphs

$$H\colon \widehat{\mathrm{Alb}}_{\ell^r,\ell^j} \longrightarrow \widehat{\mathrm{Alb}}_{\ell^{r-j},1},$$

that we denote, by slight abuse of notation, by the same symbol. This map has the following important property, where we recall that $(\underline{R}, S)$ is loopless by the assumption in this section:

Proposition 6.14. *Let $(b_s)_{s\in S}$ be a Λ-solution of $(\underline{R}, S)$ in $\mathrm{Alb}_{\ell^r,\ell^j}$. Let $\hat{b}_s \in C_1(\widehat{\mathrm{Alb}}_{\ell^r,\ell^j}, \Lambda)$ be the corresponding 1-chain obtained via refinement. Then:*

(1) The collection of 1-chains

$$H_*\hat{b}_s \in C_1(\widehat{\mathrm{Alb}}_{\ell^{r-j},1}, \Lambda)$$

with $s \in S$ is a Λ-solution of $(\underline{R}, S)$ in $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$.

(2) The color profiles of b_s and $H_*\hat{b}_s$ coincide: $\lambda(b_s) = \lambda(H_*\hat{b}_s)$.

Proof. Since

$$H\colon \widehat{\mathrm{Alb}}_{\ell^r,\ell^j} \longrightarrow \widehat{\mathrm{Alb}}_{\ell^{r-j},1}$$

is a map of S-colored graphs, and $(b_s)_{s\in S}$ is a Λ-solution, we find that for any $c_s \in \Lambda$, $s \in S$, with $\sum_s c_s e_s \in U_\Lambda$, we have $\sum_s c_s b_s \in H_1(\mathrm{Alb}_{\ell^r,\ell^j}, \Lambda)$. Hence,

$$\sum_s c_s \hat{b}_s \in H_1(\widehat{\mathrm{Alb}}_{\ell^r,\ell^j}, \Lambda).$$

We thus conclude

$$H_* \sum_s c_s \hat{b}_s = \sum_s c_s H_* \hat{b}_s \in H_1(\widehat{\mathrm{Alb}}_{\ell^{r-j},1}, \Lambda).$$

This proves that the collection of 1-chains $H_*\hat{b}_s$, $s \in S$, is a Λ-solution of $(\underline{R}, S)$ in $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$, as we want in item (1) of the proposition.

It remains to prove item (2). Note that the map $H\colon \widehat{\mathrm{Alb}}_{\ell^r,\ell^j} \to \widehat{\mathrm{Alb}}_{\ell^{r-j},1}$ contracts edges and so it is not directly suitable to compare color profiles. Instead, we argue as follows. Let $s \in S$. Since $(\underline{R}, S)$ is loopless, there is an element

$$u_s = \sum_{t\in S} c_t e_t \in U_\Lambda \qquad \text{with } c_s = 1,$$

see Lemma 2.3. Since $(b_s)_{s\in S}$ is a solution, we conclude that

$$\alpha := \sum_{t\in S} c_t b_t = b_s + \sum_{t\in S\setminus\{s\}} c_t b_t \in H_1(\mathrm{Alb}_{\ell^r,\ell^j}, \Lambda)$$

is closed. The s-color profile $\lambda_s(\alpha) = \lambda_s(b_s)$ can then be computed as follows.

Consider the composition

$$f\colon \mathrm{Alb}_{\ell^r,\ell^j} \hookrightarrow \mathcal{T}_{\ell^r,\ell^j} \xrightarrow{\xi} \mathcal{T}_{\ell^j,\ell^j} = \mathbb{R}^S/\ell^j\mathbb{Z}^S \xrightarrow{\mathrm{pr}_s} (\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s.$$

This identifies to a map of S-colored oriented graphs, where $(\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s$ denotes the graph that consists of a chain of ℓ^j edges of color s and of the same orientation, where the starting and end point of the chain are glued to give a circle. Moreover, the map f does not contract any edge of color s. Hence,

$$\lambda_s(\alpha) = \lambda_s(f_*\alpha) = \ell^j \mu,$$

where $\mu \in \Lambda$ is the unique element such that

$$f_*\alpha = \mu \cdot [(\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s] \in H_1((\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s, \Lambda),$$

and where $[(\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s]$ denotes the positively oriented generator of $H_1((\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s, \Lambda)$.

Note that

$$f_*\alpha = f_* \sum_{t\in S} c_t \hat{b}_t = f_* H_* \sum_{t\in S} c_t \hat{b}_t \in H_1((\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s, \Lambda), \tag{6.3.6}$$

where the first equality uses that $\alpha = \sum_{t\in S} c_t b_t$ and $\sum_{t\in S} c_t \hat{b}_t$ yield the same class in singular homology of $\mathcal{T}_{\ell^r,\ell^j}$, and the second equality uses that H, when viewed as a map $T_{\ell^r,\ell^j} \to \mathcal{T}_{\ell^r,\ell^j}$, is homotopic to the identity. More precisely, the second equality follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Alb}_{\ell^r,\ell^j} \arrow[r] \arrow[d, "H"] & \mathcal{T}_{\ell^r,\ell^j} \arrow[r] \arrow[d, "H"] & \mathcal{T}_{\ell^j,\ell^j} \arrow[r] \arrow[d, "h"] & (\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s \arrow[d, "h_s"] \\
\mathrm{Alb}_{\ell^r,\ell^j} \arrow[r] & \mathcal{T}_{\ell^r,\ell^j} \arrow[r] & \mathcal{T}_{\ell^j,\ell^j} \arrow[r] & (\mathbb{R}/\ell^j\mathbb{Z})\cdot e_s,
\end{tikzcd}$$

where the rightmost vertical arrow h_s is induced by $\tilde{h}$ from (6.3.1) (for $S = \{s\}$) hence homotopic to the identity, and the two horizontal compositions coincide with f. By (6.3.6) and the above description of λ_s, we then deduce that

$$\lambda_s(\alpha) = \lambda_s\left(H_* \textstyle\sum_{t\in S} c_t \hat{b}_t\right).$$

Hence,

$$\lambda_s(b_s) = \lambda_s(\alpha) = \lambda_s\left(H_* \textstyle\sum_{t\in S} c_t \hat{b}_t\right) = \lambda_s\left(H_* \hat{b}_s\right),$$

where the last equality uses that $H_* b_t$ is a 1-chain of color t, because $H\colon \widehat{\mathrm{Alb}}_{\ell^r,\ell^j} \to \widehat{\mathrm{Alb}}_{\ell^{r-j},1}$ is a map of S-colored graphs. This proves item (2) in the proposition and hence concludes the proof. $\square$

We are now in the position to finish the proof of Theorem 6.1.

Proof of Theorem 6.1. By Lemma 6.7, it suffices to produce an ℓ^i-indivisible Λ-solution in $\widehat{\mathrm{Alb}}_{\ell^{r-j},1}$ which is constant modulo ℓ^r. This is achieved by Proposition 6.14. $\square$

7. From 2-indivisible solutions to cographic matroids

In this section we prove the following, where we recall that $\mathrm{Alb}_2(\underline{R}) := \mathrm{Alb}_{2,1}(\underline{R})$:

Theorem 7.1. *Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$. Assume that $(\underline{R}, S)$ admits a 2-indivisible $\mathbb{Z}/2$-solution in $\mathrm{Alb}_2(\underline{R})$. Then $(\underline{R}, S)$ is cographic.*

7.1. Closed under taking minors.

In this section we show that having ℓ^i-indivisible Λ-solutions in $\mathrm{Alb}_{\ell^r,\ell^j}$ is a condition that is closed under taking minors.

Proposition 7.2. *Let $(\underline{R}', S')$ be a minor of a regular matroid $(\underline{R}, S)$. Let ℓ be a prime number and let $0 \le j \le i \le r$ be integers. Let Λ be a $\mathbb{Z}_{(\ell)}$-algebra. If $(\underline{R}, S)$ admits an ℓ^i-indivisible Λ-solution in $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ which is constant modulo ℓ^r, then $(\underline{R}', S')$ admits an ℓ^i-indivisible Λ-solution in $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$ which is constant modulo ℓ^r.*

Proof. Note first that the question of whether $(\underline{R}, S)$ has an ℓ^i-indivisible Λ-solution in $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ is by Lemma 2.2 independent of the chosen realization, see Lemma 5.6 and Remark 5.9. It thus suffices to prove the proposition for the integral realization $S' \to (U')^*$ of $(\underline{R}', S')$ that is induced by a given integral realization $S \to U^*$, $s \mapsto y_s$, of $(\underline{R}, S)$.

By the definition of a minor, it suffices to prove the proposition in the case where $(\underline{R}', S')$ is a deletion or a contraction of $(\underline{R}, S)$.

Case 1. $S' \subset S$ and $(\underline{R}', S')$ is the deletion of $\underline{R}$ to S'.

In this case, let $(U')^* \subset U^*$ be the span of the linear forms y_s with $s \in S'$. Then the induced integral realization of $(\underline{R}', S')$ is given by $S' \to (U')^*$, $s \mapsto y_s$. The dual of this yields an embedding $U' \hookrightarrow \mathbb{Z}^{S'}$. With respect to this realization, $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$ can be realized as the 1-skeleton

$$\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}') \subset \mathbb{R}^{S'}/(\ell^j U' + \ell^r \mathbb{Z}^{S'})$$

of $\mathbb{R}^{S'}/(\ell^j U' + \ell^r \mathbb{Z}^{S'})$, see Remark 6.4. A similar description holds for

$$\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}) \subset \mathbb{R}^{S}/(\ell^j U + \ell^r \mathbb{Z}^{S}).$$

This description shows that the natural projection map $\mathbb{Z}^S \to \mathbb{Z}^{S'}$, which maps U into U', induces a map of oriented S-colored graphs

$$\pi\colon \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}) \longrightarrow \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}'),$$

where we view the S'-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$ as an S-colored graph (without any edge of color $s \in S \setminus S'$). This map does not contract any edge of color $s \in S'$ and so $\lambda(\gamma) = \lambda(\pi_*\gamma)$ for any 1-chain γ of color $s \in S'$. It follows that if $(b_s)_{s\in S}$ is an ℓ^i-indivisible Λ-solution of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ which is constant modulo ℓ^r, then $(\pi_* b_s)_{s\in S'}$ is an ℓ^i-indivisible Λ-solution of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$ which is constant modulo ℓ^r. This concludes the proof in Case 1.

Case 2. $S' = S \setminus T$ for an independent set T of $(\underline{R}, S)$ and $(\underline{R}', S')$ is the contraction of $\underline{R}$ by T.

Consider the realization $S \to U^*$ of $\underline{R}$ and let $\langle T\rangle \subset U^*$ be the span of T. The induced realization of $\underline{R}'$ is then given by $S' \to U'^*$, where $U'^* = U^*/\langle T\rangle$ is a quotient of U^*. Dualizing this quotient map, we get an inclusion $U' \hookrightarrow U$, where

$$U' = U \cap \mathbb{Z}^{S'}$$

is the intersection of U with the subspace $\mathbb{Z}^{S'} \subset \mathbb{Z}^S$. This description shows that there is a natural embedding of Albanese tori

$$\mathbb{R}^{S'}/(\ell^j U' + \ell^r \mathbb{Z}^{S'}) \longleftrightarrow \mathbb{R}^{S}/(\ell^j U + \ell^r \mathbb{Z}^{S})$$

with associated embedding on Albanese graphs

$$\iota\colon \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}') \longhookrightarrow \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}). \tag{7.1.1}$$

Each element $[x] \in V := \mathbb{Z}^S/(\ell^j U + \ell^r \mathbb{Z}^S)$ acts via translation on $\mathbb{R}^S/(\ell^j U + \ell^r \mathbb{Z}^S)$. The corresponding action restricts to a self-map of the oriented S-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$. We may thus consider the translates of the image of ι:

$$G := \bigcup_{[x]\in V} \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}') + [x] \longhookrightarrow \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}).$$

We identify G with a subgraph of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ and note that any edge of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$ of color $s \in S'$ is contained in G. Note moreover that there is a natural map of oriented S-colored graphs

$$\pi\colon G \longrightarrow \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}'),$$

whose restriction to $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}') + [x]$ is given by translation by $[-x]$.

Let now $(b_s)_{s\in S}$ be an ℓ^i-indivisible Λ-solution of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$. We claim that the subcollection $(b_s)_{s\in S'}$ is an ℓ^i-indivisible Λ-solution of $\underline{R}'$ in the S'-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$. To prove this, let $\sum_{s\in S'} c_s e_s \in U'_\Lambda$. We then have

$$\sum_{s\in S'} c_s e_s = \sum_{s\in S'} c_s e_s + \sum_{t\in T} 0 \cdot e_t \in U_\Lambda.$$

Since $(b_s)_{s\in S}$ is a Λ-solution of $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R})$, we find that the 1-chain

$$\sum_{s\in S'} c_s b_s \in C_1(\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}), \Lambda)$$

is closed. This is a sum of 1-chains of colors contained in S', hence it is supported on the subgraph G above. We may then consider the map $\pi\colon G \to \mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$ from above and we find that the collection of 1-chains $(\pi_* b_s)_{s\in S'}$ on $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$ is a Λ-solution of $\underline{R}'$ in the S'-colored graph $\mathrm{Alb}_{\ell^r,\ell^j}(\underline{R}')$. This solution is ℓ^i-indivisible and constant modulo ℓ^r because the same holds for $(b_s)_{s\in S}$ by assumption and because π is a map of oriented S-colored graphs that does not contract any edge. This concludes the proof of the proposition. □

7.2. **Reduced Albanese graph and excluded minors.** Recall from Remark 5.5 that the oriented S-colored graph $\mathrm{Alb}_2(\underline{R}) = \mathrm{Alb}_{2,1}(\underline{R})$ has the property that for each edge of color s which points from a vertex v to another vertex w, there is also an edge of the same color between the same vertices which points in the other direction. For computations with $\mathbb{F}_2 = \mathbb{Z}/2$-homology, orientations do not play any role and so it will be convenient to contract these multiple edges, as follows.

Definition 7.3. Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$. The reduced Albanese graph $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$ is the unique unoriented S-colored graph that admits a morphism of S-colored graphs $\pi\colon \mathrm{Alb}_2(\underline{R}) \to \mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$ which is an isomorphism on edges and which identifies parallel edges of the same color to a single edge.

Since orientations play no role when working with $\mathbb{F}_2$-coefficients, we may for any S-colored graph G, such as $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$, define color profile maps $\lambda\colon C_1(G, \mathbb{F}_2) \to \mathbb{F}_2^S$ as well as $\mathbb{F}_2$-solutions in G (and 2-indivisible $\mathbb{F}_2$-solutions) analogous to Definitions 5.1 and 5.8.

Lemma 7.4. *Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$. If $\underline{R}$ admits a 2-indivisible $\mathbb{F}_2$-solution in $\mathrm{Alb}_2(\underline{R})$, then it also admits a 2-indivisible $\mathbb{F}_2$-solution in $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$.*

Proof. This follows from the fact that we may pushforward any 2-indivisible $\mathbb{F}_2$-solution in $\mathrm{Alb}_2(\underline{R})$ via the map of S-colored graphs $\pi\colon \mathrm{Alb}_2(\underline{R}) \to \mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$ (which does not contract any edge) to get a 2-indivisible $\mathbb{F}_2$-solution in $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$. □

Recall the complete graph K_5 on 5 vertices with $\binom{5}{2} = 10$ edges, and the utility graph $K_{3,3}$ with 6 vertices and $3 \cdot 3 = 9$ edges. In what follows, we say that an $\mathbb{F}_2$-solution $(b_s)_{s\in S}$ in an S-colored graph G is 2-*divisible*, if $\lambda(b_s) = 0 \in \mathbb{F}_2^S$ for all $s \in S$.

Proposition 7.5. *Let $(\underline{R}, S)$ be the graphic matroid $M(K_5)$ or $M(K_{3,3})$. Then $(\underline{R}, S)$ does not admit a 2-indivisible $\mathbb{F}_2$-solution in $\mathrm{Alb}_2(\underline{R})$ or $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$. In fact, any $\mathbb{F}_2$-solution $(b_s)_{s\in S}$ in $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R})$ is 2-divisible.*

Proof. By Lemma 5.6, the space of ℓ-indivisible $\mathbb{F}_\ell$-solutions in $\mathrm{Alb}_\ell(\underline{R})$ is independent of the chosen realization $S \to U^*$. For a regular matroid of rank g on n elements, the graph Alb_ℓ has ℓ^{n-g} vertices and $n \cdot \ell^{n-g}$ edges. Ranging over a g element basis of U, condition (2) in Definition 5.8 amounts to $g \cdot \ell^{n-g}$ conditions on the $\mathbb{F}_\ell$-coefficients of the edges. Thus, the space of $\mathbb{F}_\ell$-solutions of $(\underline{R}, S)$ in its Albanese graph is the right kernel of a $g \cdot \ell^{n-g} \times n \cdot \ell^{n-g}$ matrix over $\mathbb{F}_\ell$. Those solutions which are ℓ-divisible, i.e. lie in $\ker(\lambda)$, modulo ℓ, are the right kernel of an augmented, $(g \cdot \ell^{n-g} + n) \times n \cdot \ell^{n-g}$ matrix.

Thus, to prove that $(\underline{R}, S)$ does not admit an ℓ-indivisible $\mathbb{F}_\ell$-solution in $\mathrm{Alb}_\ell(\underline{R})$, it suffices to verify that the matrix and its augmentation have the same rank. For Alb_2, Lemma 7.4 allows us to further reduce the computational difficulty by considering $\mathrm{Alb}_2^{\mathrm{red}}$ whose solutions correspond to a submatrix with half the columns (i.e. edges) as Alb_2.

$M(K_{3,3})$ and $M(K_5)$ are, respectively, matroids of rank $g = 5$ on $n = 9$ elements, and rank $g = 4$ on $n = 10$ elements. Thus, the space of solutions for $\mathrm{Alb}_2^{\mathrm{red}}$ are kernels of 80×72 and 256×320 matrices over $\mathbb{F}_2$ respectively, while the 2-divisible solutions are kernels of matrices of size 89×72 and 266×320, respectively.

Computing the kernel in SAGE [Sag25; EGFS25c], one finds that indeed, all solutions are 2-divisible—these kernels are, respectively, of rank 15 and 103 over $\mathbb{F}_2$ for $M(K_{3,3})$ and $M(K_5)$. This proves the proposition. ☐

Proposition 7.6. *The* $\underline{R}_{10}$ *matroid does not admit* 2*-indivisible* $\mathbb{F}_2$*-solutions in* $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R}_{10})$*. In fact, any* $\mathbb{F}_2$*-solution is* 2*-divisible.*

Proof. The first claim follows from Proposition 7.2, the fact that any 1-element deletion of $\underline{R}_{10}$ is $M(K_{3,3})$, and Proposition 7.5. One may also directly compute that all $\mathbb{F}_2$-solutions in $\mathrm{Alb}_2^{\mathrm{red}}(\underline{R}_{10})$ are 2-divisible (the rank of the solution space over $\mathbb{F}_2$ is 35). ☐

7.3. Proof of Theorem 7.1.

Proof of Theorem 7.1. By a theorem of Tutte, see [Oxl92, p. 441, Corollary 13.3.4], a regular matroid is cographic if and only if it does not have the graphic matroid associated to K_5 or $K_{3,3}$ as a minor. The theorem thus follows from Propositions 7.2 and 7.5. ☐

8. Conclusions, applications, and discussion

8.1. Quadratic splittings in cographic matroids.

Proof of Theorem 1.8. Let B and $B^\star = B \setminus H$ be as in Section 4.1 and let $\pi^\star \colon X^\star \to B^\star$ be a matroidal family of principally polarized abelian varieties associated to a regular matroid $(\underline{R}, S)$ with integral realization $S \to U^*$, see Definition 4.1. Let $\iota \colon C_t \to X_t$ be a non-constant morphism from a curve C_t to a very general fiber X_t of $\pi^\star$ with $\iota_*[C_t] = m[\Theta]^{g-1}/(g-1)!$. Let ℓ be a prime that is coprime to m. Note that there is an isomorphism of fields $\overline{\mathbb{C}(B)} \cong \mathbb{C}$ under which the geometric generic fiber of $\pi^\star$ is identified with X_t. Hence, the geometric generic fiber of $\pi^\star$ contains a curve whose cohomology class is an ℓ-prime multiple of the minimal class. (Alternatively, this conclusion can also be derived from a standard Hilbert scheme argument.) It thus follows from Theorem 4.10 and Remark 4.8 that there is a positive integer d, such that the regular matroid $\underline{R}$ admits a quadratic $\mathbb{Z}_{(\ell)}$-splitting of level d into a cographic matroid. ☐

Proof of Theorem 1.9. Clearly, any cographic matroid admits, for any ring Λ and any positive integer d, a Λ-splitting of level d into a cographic matroid. Conversely, let $(\underline{R}, S)$ be a regular matroid which admits a $\mathbb{Z}_{(2)}$-splitting of some level d in a cographic matroid associated to a graph G. Up to performing some deletions, we may assume that $(\underline{R}, S)$ is loopless. Moreover, by Lemma 4.9, we may without loss of generality assume that $(\underline{R}, S)$ admits a $\mathbb{Z}_{(2)}$-splitting of level d in a graph (see Definition 4.6). It thus follows from Theorems 5.10, 6.1, and 7.1 that $\underline{R}$ is cographic. Here, we used that any 2-indivisible $\mathbb{Z}_{(2)}$-solution in $\mathrm{Alb}_2(\underline{R})$ naturally gives rise to a 2-indivisible $\mathbb{Z}/2$-solution in $\mathrm{Alb}_2(\underline{R})$. This concludes the proof of the theorem. ☐

Proof of Theorem 1.10. Let $(\underline{R}, S)$ be a regular matroid of rank g. By [EGFS25a, Proposition 4.10], we may then construct a family $\pi^\star\colon X^\star \to B^\star$ of principally polarized abelian varieties of dimension g which, with respect to an snc extension $B^\star \subset B$ and a point $0 \in B \setminus B^\star$ in the boundary, is a matroidal family associated to $\underline{R}$, see Definition 4.1. The very general fiber of $\pi^\star$ has the property that $(g-1)!$ times its minimal class is clearly represented by an algebraic curve (namely by the self-intersection of Θ). It thus follows from Theorem 1.8 that, for any ring Λ in which $(g-1)!$ is invertible, $(\underline{R}, S)$ admits a Λ-splitting of some level d into a graph, and hence, by Lemma 4.9, into a cographic matroid. This concludes the proof of the theorem. □

8.2. **Curves on very general fibers of matroidal families.**

Proof of Theorem 1.6. The first paragraph of Section 1.5 proves Theorem 1.6 as a formal consequence of Theorems 1.8 and 1.9, proven above. □

Remark 8.1. Recall from Remarks 4.2 and 4.3, that for any regular matroid $\underline{R}$ of rank g on an n-element ground set S, there exists a matroidal family of g-dimensional principally polarized abelian varieties over an n-dimensional base that is associated to $\underline{R}$, cf. [EGFS25a, Proposition 4.10]. Theorem 1.6 then shows that if $\underline{R}$ is not cographic, then the very general fiber of such a family does not contain a curve whose cohomology class is an odd multiple of the minimal class. In particular, the integral Hodge conjecture fails for the very general fiber, see Lemma 2.8.

Corollary 8.2. *Let B be a smooth quasi-projective variety and let $H \subset B$ be an snc divisor which restricts to the coordinate hyperplanes on an embedded polydisc $\Delta^S \subset B$, centered at a distinguished point $0 \in B$. Let $B^\star = B \setminus H$ and let $\pi^\star\colon X^\star \to B^\star$ be a matroidal family of principally polarized abelian varieties, associated to a regular matroid $\underline{R}$. If the restriction $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ to the punctured polydisc does not analytically deform to a family of Jacobians of curves, the integral Hodge conjecture fails for X_t.*

Proof. The matroidal information is encoded in the monodromy and thus depends only on the homeomorphism type of the real torus bundle over $(S^1)^S$ associated to the family $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$. Hence, if the restriction $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ to the punctured polydisc deforms to a family of Jacobians of curves, then $\underline{R}$ is cographic. The converse follows from [EGFS25a, Remark 2.31] and the fact that any cographic matroid can be realized via a matroidal family of Jacobians of curves, see e.g. [EGFS25a, Example 4.13]. Hence, the condition that $X^\star_{(\Delta^\star)^S} \to (\Delta^\star)^S$ does not deform to a family of Jacobians of curves is equivalent to asking that $\underline{R}$ is not cographic and so the corollary follows from Theorem 1.6 and Lemma 2.8. □

8.3. **Application to IHC for abelian varieties.**

Proof of Theorem 1.1. Let (X, Θ) be a very general principally polarized abelian variety of dimension $g \geq 4$ and let $Z \subset X$ be an equidimensional closed subscheme of codimension c with $2 \leq c \leq g-1$. Since X is very general, the Mumford–Tate group of X is maximal and so there is some non-negative integer m with

$$[Z] = m \cdot [\Theta]^c/c! \in H^{2c}(X, \mathbb{Z}).$$

To prove the theorem, it suffices to show that m is even.

Let us first assume that $g = 4$ and $c = 3$. Let $(\underline{R}, S)$ be the graphic matroid $M(K_5)$ with integral realization $S \to \mathbb{Z}^S/H_1(K_5, \mathbb{Z}) = U^*$, where S denotes the set of edges of K_5, equipped with some orientation. Then S has 10 elements, and $\operatorname{rank}(\underline{R}) = \dim U = 4$. By [EGFS25a, Proposition 4.10], there is a matroidal family $\pi^\star \colon X^\star \to B^\star$ of principally polarized abelian fourfolds associated to $\underline{R}$, cf. Definition 4.1 and Remarks 4.2 and 4.3. Since K_5 is not planar, $M(K_5)$ is not cographic. It thus follows from Theorem 1.6 that the very general fiber X_t of $\pi^\star$ does not contain a curve whose cohomology class is an odd multiple of the minimal class. This proves the case $g = 4$ and $c = 3$, because X specializes to X_t. (In fact, X_t is a very general principally polarized abelian variety of dimension 4, because $\dim B^\star = 10$ and one can check that the moduli map $B^\star \to \mathcal{A}_4$ is dominant.) The case $g = 4$ and $c = 2$ follows from the simple observation that $[\Theta] \cdot [\Theta]^2/2 = [\Theta]^3/2$ is an odd multiple of the minimal curve class on X. This proves the theorem for $g = 4$.

Let now $g \geq 5$ and let (X, Θ) be a very general principally polarized abelian variety of dimension g. We specialize (X, Θ) to a product $(Y, \Theta_Y) \times (E, \Theta_E)$ where E is an elliptic curve and (Y, Θ_Y) is a very general principally polarized abelian variety of dimension $g-1$. Let $p : Y \times E \to Y$ and $q : Y \times E \to E$ be the projections. Let further $Z_0 \subset Y \times E$ be the specialization of $Z \subset X$. Then we find that

$$[Z_0] = \frac{m}{c!} \cdot p^*[\Theta_Y]^c + \frac{m}{(c-1)!} \cdot p^*[\Theta_Y]^{c-1} \cup q^*[\Theta_E] \in H^{2c}(Y \times E, \mathbb{Z}).$$

Hence,

$$p_*[Z_0] = m[\Theta_Y]^{c-1}/(c-1)! \in H^{2c-2}(Y, \mathbb{Z}) \qquad \text{and} \qquad \iota^*[Z_0] = m[\Theta_Y]^c/c! \in H^{2c}(Y, \mathbb{Z}),$$

where $\iota \colon Y \hookrightarrow Y \times E$ denotes the inclusion. Since $2 \leq c \leq g-1$ and (Y, Θ_Y) is very general of dimension $g-1$, we conclude by induction that m is even. This concludes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.2. Let (X, Θ) be a very general principally polarized abelian fourfold. By Theorem 1.1, it suffices to show that $2[\Theta]^c/c!$ is algebraic for $c = 2, 3$. This is clear for $c = 2$ and follows for $c = 3$ from the fact that (X, Θ) is a Prym variety. $\square$

8.4. **Application to cubic threefolds.**

Proof of Theorem 1.3. Consider the Segre cubic threefold

$$Y_0 := \left\{ \sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0 \right\} \subset \mathbb{P}^5,$$

which is the unique cubic threefold with 10 nodes. Let B be a smooth quasi-projective variety with an embedded polydisc $\Delta^{10} \subset B$ and an snc divisor $H \subset B$ that restricts to the coordinate hyperplanes on Δ^{10}. Choose B such that there is a flat family of cubic threefolds $Y \to B$ which is smooth over $B^\star = B \setminus H$ and such that the fiber over 0 is the Segre cubic Y_0 from above. Assume moreover that $Y \to B$ has maximal variation in the sense that the restriction $Y_{\Delta^{10}} \to \Delta^{10}$ is a universal deformation of Y_0 (note that $\mathrm{Def}_{Y_0} = \Delta^{10}$ and such deformations Y exist, for instance, over a suitable étale open neighborhood of $0 \in \mathbb{A}^{10}$). Let $\pi^\star \colon X^\star := JY^\star \to B^\star$ be the associated smooth projective family of intermediate Jacobians.

It follows from a result of Gwena [Gwe04] that $\pi^\star$ is a matroidal family, associated to the matroid $\underline{R}_{10}$, see also [EGFS25a, Examples 2.16 and 4.7]. Since $\underline{R}_{10}$ is not cographic, we conclude from Theorem 1.6 that the very general fiber JY_t of $JY^\star \to (\Delta^\star)^{10}$ does not contain a curve whose cohomology class is an odd multiple of the minimal class. This concludes the proof of the theorem, because JY_t has Picard rank 1. □

Proof of Corollary 1.4. By Theorem 1.3, the very general cubic threefold Y has the property that the minimal curve class of its intermediate Jacobian JY is not algebraic. This implies by [Voi17] that Y does not admit a decomposition of the diagonal. Hence, Y is neither stably nor retract rational, nor $\mathbb{A}^1$-connected, see for instance [LS24] and the references therein. □

Proof of Corollary 1.5. Let X and Y be abelian varieties and let $f \colon X \times Y \to \prod_i JC_i$ be an isogeny to a product of Jacobians of curves. Assume that f has odd degree. Then $f_* \colon H_2(X \times Y, \mathbb{Z}_{(2)}) \to H_2(\prod_i JC_i, \mathbb{Z}_{(2)})$ induces an isomorphism. By [BGF23], the integral Hodge conjecture holds for curves classes on $\prod_i JC_i$. Since $H_2(X, \mathbb{Z}_{(2)})$ admits a split embedding in $H_2(X \times Y, \mathbb{Z}_{(2)})$, it follows that any $\mathbb{Z}_{(2)}$-linear combination of Hodge classes in $H_2(X, \mathbb{Z})$ is a $\mathbb{Z}_{(2)}$-linear combination of algebraic classes. In the special case where X is a g-dimensional principally polarized abelian variety with theta divisor Θ, this implies that an odd multiple of $[\Theta]^{g-1}/(g-1)!$ is algebraic. The corollary thus follows from Theorems 1.1 and 1.3. □

8.5. **Discussions and open problems.**

Definition 8.3. Let ℓ be a prime number. We define $\mathcal{M}_\ell$ as the class of matroids consisting of all regular matroids $(\underline{R}, S)$ that admit a $\mathbb{Z}/\ell$-solution $(b_s)_{s\in S}$ in $\mathrm{Alb}_\ell(\underline{R})$ with constant nonzero color profile $\lambda(b_s) = c \in (\mathbb{Z}/\ell)^*$ for all $s \in S$.

Remark 8.4. By Proposition 7.2, $\mathcal{M}_\ell$ is closed under taking minors.

Remark 8.5. By Lemma 4.9 together with Theorems 5.10 and 6.1, we see that any regular matroid $\underline{R}$ that admits a $\mathbb{Z}_{(\ell)}$-splitting of some level d in a cographic matroid is contained in $\mathcal{M}_\ell$. Theorem 1.10 thus implies that $\underline{R} \in \mathcal{M}_\ell$ for all $\ell \geq g$, where g denotes the rank of $\underline{R}$.

Remark 8.6. If ℓ is odd, then one can check that $M(K_5)$ and $M(K_{3,3})$ are contained in $\mathcal{M}_\ell$ and so is, for instance, $\underline{R}_{10}$. Indeed, by Theorem 1.9 and Remark 8.5, it suffices to prove these claims for the prime $\ell = 3$, which can be done via a similar computation as in Proposition 7.5 and 7.6. Alternatively, one can observe that the relevant matroidal families of principally polarized abelian varieties can be realized as Prym varieties and so the Prym curve yields a curve whose cohomology class is twice the minimal class. Using this we can argue as in the proof of Theorem 1.9 to conclude.

We have just proven that $\mathcal{M}_2$ is the class of cographic matroids; it can be characterized by Tutte's theorem via two excluded minors: the graphic matroids associated to K_5 and $K_{3,3}$. By the Robertson-Seymour theorem and its expected extension to regular matroids, the class $\mathcal{M}_\ell$ of regular matroids is by Proposition 7.2 expected to be determined by a finite list of excluded minors. This leads to the following natural open problem.

Problem 8.7. Let ℓ be an odd prime. Characterize the class $\mathcal{M}_\ell$ of regular matroids via a finite list of excluded minors.

Definition 8.8. Let $(\underline{R}, S)$ be a regular matroid. The *radical distance* of $(\underline{R}, S)$ to the class of cographic matroids is the integer

$$d(\underline{R}) := \mathrm{lcm}\{\ell \mid \underline{R} \notin \mathcal{M}_\ell\}$$

By Theorem 7.1, $d(\underline{R}) = 1$ if and only if $\underline{R}$ is cographic. By Remark 8.5 (resp. Theorem 1.10), $d(\underline{R})$ is always finite and universally bounded by the product of all primes which are smaller than $\mathrm{rank}(\underline{R})$.

Some of the main results of this paper may then be summarized as follows.

Theorem 8.9. *Let $(\underline{R}, S)$ be a regular matroid. Let $\pi^\star \colon X^\star \to B^\star$ be a matroidal family of principally polarized abelian varieties associated to a $(\underline{R}, S)$, see Definition 4.1 and Remark 4.3. Let $C_t \subset X_t$ be a curve on a very general fiber of $\pi^\star$ with*

$$[C_t] = m \cdot [\Theta]^{g-1}/(g-1)! \in H_2(X_t, \mathbb{Z}).$$

Then m is divisible by $d(\underline{R})$.

Proof. By the Chinese remainder theorem, it suffices to show $m \equiv 0 \bmod \ell$ for all primes ℓ with $\underline{R} \notin \mathcal{M}_\ell$. For a contradiction, assume $\underline{R} \notin \mathcal{M}_\ell$ and m is coprime to ℓ. Then, Theorems 4.10, 5.10, and 6.1 show that $\underline{R} \in \mathcal{M}_\ell$, which contradicts our assumptions. □

Proposition 8.10. *Let $K_{3,5}$ be the complete bipartite graph on $3+5=8$ vertices. Then the graphic matroid $M(K_{3,5})$ is an excluded minor for the class $\mathcal{M}_3$. In particular, for $g \geq 7$, the minimal multiple of the minimal class which is algebraic on a very general principally polarized abelian g-fold is at least 6.*

Proof. From the computations in [EGFS25c], $M(K_{3,5}) \notin \mathcal{M}_3$ and any 1-element deletion or contraction of $M(K_{3,5})$ lies in $\mathcal{M}_3$. Thus $M(K_{3,5})$ is an excluded minor for the class $\mathcal{M}_3$. Since $\operatorname{rank} M(K_{3,5}) = 7$, the proposition follows from Theorem 8.9 and Proposition 7.2; indeed $M(K_{3,5})$ is a minor of $M(K_{3,n})$ (of rank $2+n$), for all $n \geq 5$. (Alternatively, one can use a specialization/induction argument, as in the proof of Theorem 1.1, to reduce to the case $g = 7$.) □

Remark 8.11. In forthcoming work [EGFS25b], we prove the same result for principally polarized abelian 6-folds, where one also has an upper bound of 6 by [ADFIO20].

We conclude with the following natural combinatorial problems.

Problem 8.12. Are there regular matroids $\underline{R}$ whose radical distance $d(\underline{R})$ from the class of cographic matroids is arbitrarily large?

Problem 8.13. What is (the asymptotic behavior of) the function

$$g \mapsto d(g) := \operatorname{lcm}\{d(\underline{R}) \mid \underline{R} \text{ is a regular matroid of rank } g\}?$$

Note that Problem 8.12 asks precisely whether $d(g)$ is unbounded in g. The interest in the function $d(g)$ lies in the following result:

Theorem 8.14. *Let $C \subset X$ be a curve on a very general principally polarized abelian variety (X, Θ) of dimension g. Then $[C] = m \cdot [\Theta]^{g-1}/(g-1)!$ for an integer m divisible by $d(g)$.*

Proof. This follows from Theorem 8.9 together with the existence of matroidal families of g-dimensional principally polarized abelian varieties, associated to any regular matroid $\underline{R}$ of rank g, see [EGFS25a, Proposition 4.10]. □

References

[ADFIO20] V. Alexeev et al. "The uniformization of the moduli space of principally polarized abelian 6-folds". In: *J. Reine Angew. Math.* 761 (2020), pp. 163–217.

[AH62] M. Atiyah and F. Hirzebruch. “Analytic cycles on complex manifolds”. In: *Topology* 1 (1962), pp. 25–45.

[ALT20] K. Adiprasito, G. Liu, and M. Temkin. “Semistable reduction in characteristic 0”. In: *Séminaire Lotharingien de Combinatoire* 82B (2020), Art. 25, 10.

[Arb+85] E. Arbarello et al. *Geometry of Algebraic Curves. Volume I*. Vol. 267. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985, pp. xvi+386.

[BC77] C. Barton and H. Clemens. “A result on the integral Chow ring of a generic principally polarized complex Abelian variety of dimension four”. In: *Compositio Math.* 34.1 (1977), pp. 49–67.

[BGF23] T. Beckmann and O. de Gaay Fortman. “Integral Fourier transforms and the integral Hodge conjecture for one-cycles on abelian varieties”. In: *Compositio Mathematica* 159.6 (2023), pp. 1188–1213.

[CG72] H. Clemens and P. Griffiths. “The intermediate Jacobian of the cubic threefold”. In: *Annals of Mathematics. Second Series* 95 (1972), pp. 281–356.

[Cle77] H. Clemens. “Degeneration of Kähler manifolds”. In: *Duke Mathematical Journal* 44.2 (1977), pp. 215–290.

[EGFS25a] P. Engel, O. de Gaay Fortman, and S. Schreieder. *Combinatorics and Hodge theory of degenerations of abelian varieties: A survey of the Mumford construction*. 2025. arXiv: 2507.15695 [math.AG]. URL: https://arxiv.org/abs/2507.15695.

[EGFS25b] P. Engel, O. de Gaay Fortman, and S. Schreieder. *Optimality of the Prym–Tyurin construction for $\mathcal{A}_6$*. To appear. 2025.

[EGFS25c] P. Engel, O. de Gaay Fortman, and S. Schreieder. *Solutions in Albanese graphs of regular matroids*. 2025. URL: https://github.com/philip-engel/regular_matroids.

[GFS25] Olivier de Gaay Fortman and Stefan Schreieder. “Abelian varieties with no power isogenous to a Jacobian”. In: *Compositio Mathematica* 161.6 (2025), 1404–1457.

[Gwe04] T. Gwena. “Degenerations of Prym Varieties and cubic threefolds”. PhD thesis. University of Georgia, 2004.

[Har77] R. Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Hir68] H. Hironaka. “Smoothing of algebraic cycles of small dimensions”. In: *Amer. J. Math.* 90 (1968), pp. 1–54.

[Kol92] J. Kollár. “Trento examples”. In: *Classification of Irregular Varieties*. Ed. by E. Ballico, F. Catanese, and C. Ciliberto. Vol. 1515. Lecture Notes in Mathematics. Springer, 1992, pp. 134–139.

[LS24] J. Lange and S. Schreieder. *On the rationality problem for hypersurfaces*. 2024. arXiv: 2409.12834 [math.AG]. URL: https://arxiv.org/abs/2409.12834.

[Mar25] E. Markman. *Cycles on abelian 2n-folds of Weil type from secant sheaves on abelian n-folds*. 2025. arXiv: 2502.03415 [math.AG]. URL: https://arxiv.org/abs/2502.03415.

[Oxl92] J. Oxley. *Matroid Theory*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992, pp. xii+532.

[Sag25] Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.6)*. https://www.sagemath.org. 2025.

[Voi17] C. Voisin. “On the universal CH_0 group of cubic hypersurfaces”. In: *Journal of the European Mathematical Society (JEMS)* 19.6 (2017), pp. 1619–1653.

[Voi24] C. Voisin. “Cycle classes on abelian varieties and the geometry of the Abel-Jacobi map”. In: *Pure and Applied Mathematics Quarterly* 20.5 (2024), pp. 2469–2496.

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