COMPLEX STRUCTURES ON $\mathbb{S}^6$

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Abstract. A recent manuscript (https://alpo.ge/s6.pdf) written by Claude, and communicated by Levent Alpöge, constructs a 1-parameter family of compact complex threefolds diffeomorphic to S^6. The goal of these notes is to describe the geometric ideas behind the construction, and give a rough indication of the proof.

Contents

1. Initial construction

Let $\pi : S \to \mathbb{P}^1$ be a rational elliptic surface whose j-map has degree 1, so that in suitable coordinates, $j = 1728t$, and for which there are three singular fibers, of Kodaira types IV^*, III, I_1. These singular fibers correspond respectively to the order 3, 4, ∞ corners of the $(3, 4, \infty)$ triangle. There is a zero section O, and the Mordell–Weil group is infinite cyclic. Let $P \in \mathrm{MW}(S)$ be a generator. The section P passes through non-identity components on both the IV^* and III fiber, whose component groups are $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively.

Consider the line bundle $M = \mathcal{O}_S(P - O)$. Translation by $6P$ defines an automorphism $t_{6P} : S \to S$. Since 2 and 3 both divide 6, the automorphism t_{6P} acts in a manner preserving the components of the IV^* and III Kodaira fibers.

Lemma 1.1. $\mathcal{H}om(t_{6P}^* M, M) \simeq \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$.

Proof. For a general fiber S_t over $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, we have that $M_t \in \mathrm{Pic}^0(S_t)$. It follows that $t_{6P}^* M_t \simeq M_t$ for all $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Hence

$t_{6P}^*M \otimes M^{-1}$ is expressible as the line bundle associated to a linear combination of vertical divisors. Since t_{6P} acts by an element of Aut^0 of every fiber, the line bundle $L := \mathcal{H}om(t_{6P}^*M, M)$ has multidegree 0 on every fiber. Thus, $L \simeq \pi^*\mathcal{O}_{\mathbb{P}^1}(n)$ for some integer n, seeing as π has no multiple fibers.

We compute n via Shioda's height pairing. Since S is rational elliptic, we have $\chi(\mathcal{O}_S) = 1$, $O^2 = -1$, and $P \cdot O = 0$. The local contributions at the IV^* and III fibers are $\frac{4}{3}$ and $\frac{1}{2}$, respectively. Hence

$$\langle P, P\rangle = 2 - \tfrac{4}{3} - \tfrac{1}{2} = \tfrac{1}{6}.$$

Consider the section $Q = 6P$. By bilinearity,

$$\langle Q, Q\rangle = 36\langle P, P\rangle = 6.$$

Since Q meets the identity component of every reducible fiber, all local contributions involving Q vanish, and Shioda's height formula gives

$$6 = \langle Q, Q\rangle = 2 + 2(Q \cdot O).$$

It follows that $(6P) \cdot O = Q \cdot O = 2$. Similarly, $\langle P, Q\rangle = 6\langle P, P\rangle = 1$ and in turn $P \cdot (6P) = P \cdot Q = 2$.

Finally, restricting $L = t_{6P}^*M^{-1} \otimes M$ to the zero section gives

$$\begin{aligned} n &= L \cdot O \\ &= (O - P) \cdot (6P) - (O - P) \cdot O \\ &= (2 - 2) - ((-1) - 0) = 1. \end{aligned}$$

□

Corollary 1.2. *There is, up to scale, a unique section*

$$\phi \in \mathcal{H}om(t_{6P}^*M, M)(S)$$

which over $\mathbb{A}^1$ is an isomorphism, i.e. a linearization of the action of t_{6P} on M, and having a simple zero along the fiber over $\infty \in \mathbb{P}^1$.

Let $M^\times := M \setminus \{\text{zero section}\}$. It is a threefold, which is a principal $\mathbb{G}_m$-bundle over the rational elliptic surface S. Let $S^\circ := S \setminus S_\infty$ and $M^{\times,\circ} := M^\times|_{S^\circ}$.

Definition 1.3. We define a $\mathbb{Z}$-action on $M^{\times,\circ}$ by acting by the translation $(t_{6P})^{-1}$ on the base S° and the linearizing isomorphism ϕ on $M^{\times,\circ}$. That is, the generator of $\mathbb{Z}$ acts by the lift of t_{6P}^{-1} defined by ϕ.

So in particular, the automorphism of $M^{\times,\circ}$ induced by ϕ lifts the automorphism t_{6P}^{-1} over S°.

Proposition 1.4. *Fix a reference linearization ϕ as in Corollary 1.2. There exists a constant c such that for all linearizations $u\phi$, $u \in \mathbb{C}^*$, with $|u| < c$, the $\mathbb{Z}$-action of Definition 1.3 is properly discontinuous.*

Proof. Consider the map $\phi\colon t_{6P}^*M \to M$ which is isomorphism away from S_∞. Choose a smooth Hermitian metric on M. With respect to this metric, the operator norm $x \mapsto \|\phi_x\|$ is a continuous function on the compact surface S, equal to zero exactly along S_∞. It is therefore bounded. Set $C = \max_{x\in S} \|\phi_x\|$ and choose $c > 0$ such that $cC < 1$.

Fix $u \in \mathbb{C}^*$ with $|u| < c$. Then, $u\phi$ also defines a self-map of M covering $(t_{6P})^{-1}$ as in Definition 1.3. Since the operator norm of $u\phi_x$ is strictly less than 1, the action is contracting with respect to the Hermitian norm function

$$M^{\times,\circ} \to \mathbb{R}_{>0}.$$

The $\mathbb{Z}$-action on this open set is therefore properly discontinuous. □

Definition 1.5. Let ϕ_0 be a linearization as in Corollary 1.2 and let $\phi = u\phi_0$ with $|u| < c$, $u \in \mathbb{C}^*$ as in Proposition 1.4. We define a smooth threefold as the quotient

$$X^\circ(u) := M^{\times,\circ}/\mathbb{Z}$$

of the $\mathbb{Z}$-action of Definition 1.3 associated to the linearization ϕ.

Proposition 1.6. *$X^\circ(u) \to \mathbb{A}^1$ is a fibration with smooth complex 2-torus fibers over $\mathbb{A}^1 \setminus \{0, 1\}$.*

Proof. Over $t \in \mathbb{A}^1 \setminus \{0, 1\}$ we have an exact sequence

$$1 \to \mathbb{C}^* \to M_t^\times \to S_t \to 0, \tag{1.0.1}$$

i.e. the fiber of $M^\times$ over t is a semiabelian variety. So the universal cover of $M_t^\times$ is $\mathbb{C}^2$, and its fundamental group is isomorphic to $\mathbb{Z}^3$.

Quotienting by the additional $\mathbb{Z}$-action associated to the linearization ϕ gives a complex 2-torus, because the action is properly discontinuous, see Proposition 1.4, and the additional $\mathbb{Z}$-factor increases the rank of the subgroup of $\mathbb{C}^2$ from 3 to 4. □

Proposition 1.7. *In a punctured neighborhood of $\infty \in \mathbb{P}^1$ with coordinate q, $X^\circ(u)$ is biholomorphic to*

$$(\mathbb{C}^*)^2/\langle(c_1, c_2q), (c_3q, c_4)\rangle$$

for holomorphic units $c_i \in \mathbb{C}^* + O(q)$.

Proof. Since S_∞ is an I_1 Kodaira fiber, the restriction of S to Δ^* admits a Tate uniformization $S_q \simeq \mathbb{C}^*/q^{\mathbb{Z}}$. After pulling $M^\times$ back to the partial cover $\mathbb{C}^* \to S_q$ and choosing a trivialization, its total space is $(\mathbb{C}^*)^2$. The deck transformation defining the Tate curve acts by

$$(z, w) \mapsto (c_3qz, c_4w), \text{ for } (z, w) \in (\mathbb{C}^*)^2.$$

Here c_3 and c_4 are holomorphic units. The first scaling factor has order one in q because it is the Tate period, while the second has order zero because M has degree zero and extends across the I_1 fiber.

The linearization ϕ gives a second transformation of $(\mathbb{C}^*)^2$. It covers translation by $-6P$ on the Tate curve. Since $6P$ specializes to a point of the smooth locus of the I_1 fiber, its multiplicative Tate coordinate is a holomorphic unit. Thus the first component of this transformation has the form $z \mapsto c_1 z$. Moreover, ϕ vanishes to first order along S_∞ so its action in the fiber direction of $M^\times$ has the form $w \mapsto c_2 q w$, where c_2 is a holomorphic unit. Hence the second transformation is

$$(z, w) \mapsto (c_1 z, c_2 q w).$$

Quotienting $(\mathbb{C}^*)^2$ by these two commuting transformations gives the asserted description of $X^\circ(u)_q$ for $q \in \Delta^*$. $\square$

Corollary 1.8. *A smooth, Kulikov filling $X^o(u) \hookrightarrow X(u)$ over $\infty \in \mathbb{P}^1$ is provided by a $\mathbb{Z}(0,1) \oplus \mathbb{Z}(1,0)$-periodic tiling of $\mathbb{R}^2$ by lattice simplices of minimal volume.*

For instance, one may take the A_2 tiling by triangles, in which case, the fiber $X(u)_\infty$ is the result of gluing the opposite sides of a hexagon of (-1)-curves on a del Pezzo surface dP_6.

Proof. This follows from a simple Mumford fan construction. The key point is that by Proposition 1.7, the fiber over the punctured polydisk is $(\mathbb{C}^*)^2$ modulo the subgroup generated by the rows of the matrix

$$c \cdot q^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \quad c = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

where $\cdot$ denotes entrywise multiplication. Thus, a $\mathbb{Z}(0,1) \oplus \mathbb{Z}(1,0) \simeq \mathbb{Z}^2$-periodic tiling of $\mathbb{R}^2$ provides a Mumford construction filling the family over $q = 0$. Taking the triangulation of the unit square into two triangles $((0,0),(0,1),(1,1))$ and $((0,0),(1,0),(1,1))$ and extending $\mathbb{Z}^2$-periodically gives such a tiling $\mathcal{T}$ of $\mathbb{R}^2$. Putting this tiling at height 1 in $\mathbb{R}^3$, i.e. $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$, and taking the cone $\mathrm{Cone}(\mathcal{T})$, we get a $\mathbb{Z}^2$-periodic fan, with standard affine cones.

A $\mathbb{Z}^2$-quotient of the toric variety of $\mathrm{Cone}(\mathcal{T})$ gives the Mumford construction, and the central fiber is the $\mathbb{Z}^2$-quotient of a periodic quilt of smooth toric surfaces, whose fans are all isomorphic to the quotient fan $\mathrm{Cone}(\mathcal{T})/\mathbb{R}_{\geq 0}(0,0,1)$, which gives dP_6. The $\mathbb{Z}^2$-quotient has the effect of taking a single such dP_6 and gluing the opposite sides of its anticanonical hexagon, as in the corollary. $\square$

Proposition 1.9. *The fibration $X(u) \to \mathbb{P}^1$ admits a holomorphic section. Over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, this section may be taken as the origin of the smooth complex 2-torus fibers.*

Proof. The restriction of M to the zero section O admits a section e which is non-vanishing over $\mathbb{A}^1$ and has a first order zero at ∞, because $M \cdot O = 1$. On a smooth fiber, e rigidifies the degree zero line bundle M_t and gives $M_t^\times$ the structure of a semiabelian group with identity e_t. More generally, e defines a section of $X^\circ(u) = M^{\times,\circ}/\mathbb{Z} \to \mathbb{A}^1$.

Near ∞, the condition that e vanishes to first order implies that it defines a local holomorphic section of the Tate uniformization of Proposition 1.7. $\square$

We have thus completed the summary of the "initial construction": We have produced a smooth, compact complex threefold $X(u)$ for all sufficiently small $u \in \mathbb{C}^*$, which is a complex 2-torus fibration, with section, and three singular fibers.

2. Logarithmic modification at 0 and 1

We now fill the family $X(u) \to \mathbb{P}^1$ at 0 and 1 in a new way. This logarithmic transform changes the filling without changing the torus family over the punctured disc.

To this end, we first describe the family $X(u)$ over a neighborhood of 0 and 1. Observe that the $\mathrm{SL}_2(\mathbb{Z})$ monodromy of $S \to \mathbb{P}^1$ about 0 and 1 have orders 3 and 4 respectively.

Proposition 2.1. *Let $\Delta_0 \ni 0$, $\Delta_1 \ni 1$ be disks. Consider the order 3, resp. 4 base change $\Delta \to \Delta_0$, $s \mapsto s^3 = t$ or $\Delta \to \Delta_1$, $s \mapsto s^4 = t$.*

1. *$S \times_{\Delta_i} \Delta$ admits a smooth elliptic reduction S' for $i = 0, 1$ isomorphic over the origin of Δ to, respectively, the triangular or square elliptic curve $E_\triangle$ or $E_\square$.*
2. *The linearization, viewed as a biholomorphism $\phi\colon M^{\times,\circ} \to M^{\times,\circ}$, lifts to biholomorphism ϕ' of $M' := \mathcal{O}_{S'}(P' - O')$ where P' and O' are the transforms of P and O.*

   Furthermore, the action of ϕ' on $E_\triangle$ or $E_\square$ is trivial, and ϕ' acts on the $\mathbb{G}_m$-torsor $M'^\times$ (which over $E_\triangle$ or $E_\square$ corresponds to either a 3-torsion or a 2-torsion line bundle $L^\times$, respectively) by a scalar in $\mathbb{C}^*$.

Proof. The smooth reduction of the IV^* and III Kodaira fibers after a monodromy-trivializing base change is well known. Let S'' be a resolution of indeterminacy of $S \times_{\Delta_i} \Delta \dashrightarrow S'$. The biholomorphism ϕ

pulls back to a biholomorphism of the $\mathbb{G}_m$-torsor $M''^{\times}$ over S'' corresponding to the pullback of M. Since t_{6P} defines an element of Aut^0 of every fiber, we deduce that the lift of ϕ to a biholomorphism of $M''^{\times}$ preserves the components contracted over the morphism $S'' \to S'$.

Hence this lift descends to a linearization ϕ' of $M'^{\times}$, which is equivariant over the action of $t_{6P'}$ on S'. By an explicit computation of S'' and the smooth model S', it is easily seen that P' intersects a 3-torsion point of $E_\triangle$ and a 2-torsion point of $E_\square$. Hence $t_{6P'}$ acts trivially on the central fiber $E_\triangle$ or $E_\square$ and the line bundle $M'|_{E_\triangle \text{ or } E_\square}$ is either a 3-torsion or 2-torsion element $L \in \mathrm{Pic}^0(E_\triangle \text{ or } E_\square)$ of the Picard group.

It follows that on the central fiber, ϕ' is simply an automorphism of the $\mathbb{G}_m$-torsor $L^\times$ (acting trivially on the base). $\square$

Corollary 2.2. *The quotient $L^\times/\mathbb{Z}$ arising from Proposition 2.1 is an abelian surface $E_\triangle \times E'/\mu_3$ or $E_\square \times E'/\mu_2$.*

Proof. As L is a 3- or 2-torsion bundle over $E_\triangle$ or $E_\square$, the pullback of L to an étale μ_3- or μ_2-cover is trivial. Along this pullback, the $\mathbb{Z}$-action lifts to an action on $\mathbb{C}^*$, whose quotient is an elliptic curve E'. Descending along the étale cover, the generator of the $\mathbb{Z}$-action and the generator of $\pi_1(\mathbb{C}^*)$ are preserved, showing that the quotient is an abelian surface (rather than bielliptic). $\square$

Corollary 2.3. *The order 3, resp. 4, base change of $X(u) \to \mathbb{P}^1$ locally near 0, resp. 1, admits a bimeromorphic modification $X' \to \Delta$ to a smooth complex 2-torus fibration, with filling as in Corollary 2.2.*

The deck group μ_3 resp. μ_4 acts on the smooth reduction $S' \to \Delta$, and we may recover $S|_{\Delta_i}$ as a relatively minimal model of the quotient. The same holds for $X(u)$ and the filling of Corollary 2.3.

Let $\Lambda \coloneqq H_1(X(u)_t, \mathbb{Z})$ be a smooth complex 2-torus fiber, for a base point $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let $\gamma_0, \gamma_1, \gamma_\infty \in \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, t)$ be simple closed curves, enclosing the corresponding point of $\mathbb{P}^1$ counterclockwise. Observe that we have the relation

$$\gamma_0\gamma_1\gamma_\infty = 1$$

and a monodromy representation

$$T_i \coloneqq \rho(\gamma_i) \in \mathrm{GL}(\Lambda) \simeq \mathrm{GL}_4(\mathbb{Z}).$$

Proposition 2.4. *We have conjugacies*

$$T_0 \sim_{\mathbb{Q}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_1 \sim_{\mathbb{Q}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_\infty \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. After the cyclic base changes of orders 3 and 4, the family has good reduction at 0 and 1. Let F_0 and F_1 denote the resulting smooth central fibers. The isogenies in Corollary 2.2 give decompositions

$$\Lambda \simeq H_1(F_i, \mathbb{Z}) \supset H_1(E_{\triangle \text{ or } \square}, \mathbb{Z}) \oplus H_1(E_i', \mathbb{Z}).$$

where the overlattice Λ is explicitly computable, of index 3 and 2 respectively. The deck transformation acts on the first summand by the usual order 3, respectively order 4, automorphism of the triangular or square elliptic curve, and acts trivially on the second summand. So

$$T_0 \sim_{\mathbb{Q}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } T_1 \sim_{\mathbb{Q}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with the $\mathbb{Z}$-conjugacy class again easily determined from the explicit overlattice Λ. At ∞, Proposition 1.7 identifies the family with the quotient of $(\mathbb{C}^*)^2$ by two periods having q-orders $(0,1)$ and $(1,0)$. So in a suitable basis,

$$T_\infty \sim_{\mathbb{Z}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

Remark 2.5. We record integral normal forms which are only slight modifications of the rational normal forms above. Bases $\mathcal{B}_0$ and $\mathcal{B}_1$ of the overlattice Λ may be chosen so that

$$[T_0]_{\mathcal{B}_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad [T_1]_{\mathcal{B}_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We may make a more detailed analysis of the topology[1] to show that the columns of the matrix

$$C = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

express $\mathcal{B}_1$ in the basis $\mathcal{B}_0$. In particular $C^2 = I$ and we have

$$[T_1]_{\mathcal{B}_0} = C[T_1]_{\mathcal{B}_1}C^{-1}.$$

[1]This is the main missing mathematical content of this note.

Since the fibration over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ has a section, these two matrices and the transition matrix C uniquely determine the topology of the real 4-torus fibration over $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

Definition 2.6. The *local invariants* are $\Lambda^{T_i} = \{\lambda \in \Lambda \,:\, T_i(\lambda) = \lambda\}$.

The finite group $\frac{1}{3}\Lambda^{T_0}/\Lambda^{T_0}$ consists of the 3-torsion translations on the base change $X(u) \times_{\Delta_0} \Delta$, which are invariant under the μ_3 deck action, while the finite group $\frac{1}{4}\Lambda^{T_1}/\Lambda^{T_1}$ consists of the 4-torsion translations on the base change $X(u) \times_{\Delta_1} \Delta$, which are invariant under the μ_4-deck action. These groups of torsion sections play an important role in the following construction.

Construction 2.7 (Log transform)**.** Given $\overline{a}_0 \in \frac{1}{3}\Lambda^{T_0}/\Lambda^{T_0}$ we may define a twisted μ_3-action $\widetilde{\rho}$ on the smooth reduction $X' \to \Delta$ of the base change $X(u) \times_{\Delta_0} \Delta$ by setting

$$\widetilde{\rho}(x) = \rho(x) + \overline{a}_0.$$

Here ρ is the generator of the original deck action of μ_3 coming from the base change $\Delta \to \Delta_0$. We may then form the twisted quotient

$$X'/\langle\widetilde{\rho}\rangle.$$

We require the following lemma:

Lemma 2.8. *The twisted quotient $X'/\langle\widetilde{\rho}\rangle$ is biholomorphic over the punctured neighborhood Δ_0^* to $X(u)|_{\Delta_0^*}$. Furthermore, a lift of $\overline{a}_0$ to some $a_0 \in \frac{1}{3}\Lambda^{T_0}$ determines explicitly such a biholomorphism.*

Proof. Put $v_0 = 3a_0 \in \Lambda^{T_0}$ and choose the generator of the deck group so that it acts on the base by $s \mapsto \zeta_3 s$. Over Δ^* define the section

$$\sigma_{a_0}(s) = \frac{\log s}{2\pi i} v_0 = \frac{3 \log s}{2\pi i} a_0.$$

Although $\log s$ is multivalued, moving to a new branch changes σ_{a_0} by an integral multiple of $v_0 \in \Lambda$. It therefore defines a single-valued holomorphic section of the family of complex tori over Δ^*. Let

$$\Psi_{a_0}(x, s) = (x + \sigma_{a_0}(s), s)$$

be translation by this section.

Since v_0 is fixed by T_0, analytic continuation gives

$$\sigma_{a_0}(\zeta_3 s) = T_0 \sigma_{a_0}(s) + a_0.$$

Consequently

$$\Psi_{a_0} \circ \rho = \widetilde{\rho} \circ \Psi_{a_0}.$$

Thus Ψ_{a_0} descends to a biholomorphism between the quotients by ρ and $\widetilde{\rho}$ over Δ_0^*. The quotient by ρ is $X(u)|_{\Delta_0^*}$ which proves the lemma. □

There is an exactly analogous construction given an element $\overline{a}_1 \in \frac{1}{4}\Lambda^{T_1}/\Lambda^{T_1}$ and a lift $a_1 \in \frac{1}{4}\Lambda^{T_1}$.

Definition 2.9. We define the analytic space $X(u, a_0, a_1)$ as the result of gluing the twisted quotients of Lemma 2.8 associated to $a_0 \in \frac{1}{3}\Lambda^{T_0}$ and $a_1 \in \frac{1}{4}\Lambda^{T_1}$ to $X(u)|_{\mathbb{P}^1\setminus\{0,1\}}$ of the logarithmic transforms.

Remark 2.10. By construction, $X(u, 0, 0)$ is bimeromorphic to $X(u)$.

Corollary 2.11. *When $\overline{a}_0$ and $\overline{a}_1$ are primitive 3-, resp. 4-torsion, the twisted action of $\widetilde{\rho}$ on X_i', for $i = 0, 1$ is free. Thus, the fiber of*

$$X(u, a_0, a_1) \to \mathbb{P}^1$$

is multiple, of

(1) *multiplicity 3 over 0, with reduced fiber a bielliptic surface, of Bagnera-de Franchis type $\mu_3 \times \mu_3$.*
(2) *multiplicity 4 over 1, with reduced fiber a bielliptic surface, of Bagnera-de Franchis type $\mu_2 \times \mu_4$.*

Proof. See Construction 2.7 and Corollary 2.2. □

Definition 2.12. The *global invariant covectors* are the linear functionals $\Lambda \to \mathbb{Z}$ which are invariant under $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \mathrm{GL}(\Lambda)$.

Let us comment a bit on the structure of Λ and the corresponding rank 4, weight -1 VHS on the corresponding local system. By construction, the complex 2-tori over a general point of $\mathbb{P}^1$ are $\mathbb{Z}$-quotients of the semiabelian variety $M_t^\times$ which fit into the exact sequence (1.0.1). Thus we have a natural filtration:

$$0 \to \Lambda^\times \to \Lambda \xrightarrow{\psi} \mathbb{Z} \to 0$$

where $\Lambda^\times = \pi_1(M_t^\times)$ and a further exact sequence

$$0 \to \mathbb{Z} \to \Lambda^\times \to \Lambda' \to 0$$

where Λ' is the rank 2, weight -1 local system underlying the $\mathbb{Z}$-VHS of the elliptic fibration $S \to \mathbb{P}^1$.

Lemma 2.13. *The space of global invariant covectors has rank 1, generated by the map $\psi \colon \Lambda \to \mathbb{Z}$.*

Sketch. This is relatively clear geometrically. First Λ' admits no invariant covectors. Then $\Lambda^\times$ will also admit no invariant covectors so long as the extension of Λ' by $\mathbb{Z}$ is sufficiently non-trivial. Roughly, this holds because $M = \mathcal{O}_S(P - O)$ is not torsion on the fibers. A precise proof is furnished by considering the integral matrices $[T_0]_{\mathcal{B}_0}$, $[T_1]_{\mathcal{B}_1}$, and change of basis C of Remark 2.5. □

We will see the significance of this global invariant covector in the course of the proof of the main theorem.

3. Main result

Theorem 3.1. *Let $Y(u) \coloneqq X(u, a_0, a_1)$ where local monodromy invariants $a_0 \in \frac{1}{3}\Lambda^{T_0}$ and $a_1 \in \frac{1}{4}\Lambda^{T_1}$ are chosen so that:*

1. *$\overline{a}_0$ and $\overline{a}_1$ are, respectively, primitive 3- and 4-torsion,*
2. *$|12\psi(a_0 + a_1)| = 1$.*

Then for all $u \in \Delta^*$ in a sufficiently small punctured disk,

$$Y(u) \simeq_{\text{diffeo}} \mathbb{S}^6$$

is a compact complex manifold, diffeomorphic to the 6-sphere.

Proof. By Section 4, $Y(u)$ is simply connected. By Section 5, it has the integral homology of $\mathbb{S}^6$. The Hurewicz theorem implies that $Y(u)$ is a homotopy 6-sphere. Smale's generalized Poincaré theorem identifies it topologically with $\mathbb{S}^6$. Finally, the group Θ_6 of oriented smooth homotopy 6-spheres is trivial by the computation of Kervaire–Milnor. Hence $Y(u)$ is diffeomorphic to the standard 6-sphere. $\square$

4. The fundamental group

Put $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and $J = Y(u)|_U = X(u)|_U$. The section of Proposition 1.9 gives a base point in every fiber. Hence

$$\pi_1(J) = \Lambda \rtimes \pi_1(U),$$

where the action on Λ is the monodromy representation. Write $\mathcal{B}_0 = (e_1, e_2, e_3, e_4)$ for the basis of Remark 2.5, normalized so that

$$\psi(e_4) = 1, \qquad \psi(e_1) = \psi(e_2) = \psi(e_3) = 0.$$

We use the same letter ψ for its extension to $\Lambda \otimes \mathbb{Q}$. In the common basis $\mathcal{B}_0$ we have

$$[T_1]_{\mathcal{B}_0} = C[T_1]_{\mathcal{B}_1}C^{-1} = \begin{pmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, ψ is fixed by both monodromies.

Since $T_\infty = (T_0T_1)^{-1}$, the preceding matrices give

$$(T_\infty - I)\Lambda = \langle e_2, e_3 \rangle =: K_\infty.$$

These are precisely the two vanishing cycles (i.e. generators of $W_{-2}H_1$) of the Mumford construction at ∞. Thus $K_\infty \subset \pi_1(J)$ is killed by filling the fiber at ∞. Since

$$T_0 \cdot e_2 = e_1 - e_2 + e_3,$$

the normal closure of K_∞ contains e_1 as well, and is exactly

$$\ker \psi = \langle e_1, e_2, e_3 \rangle.$$

Thus the image of Λ in π_1 of the extension of J over ∞ is infinite cyclic, generated by the image c of e_4. Because ψ is monodromy invariant, c is central.

Put $v_0 = 3a_0 \in \Lambda^{T_0}$, $v_1 = 4a_1 \in \Lambda^{T_1}$, and set

$$m = \psi(v_0) = 3\psi(a_0) \in \mathbb{Z}, \qquad n = \psi(v_1) = 4\psi(a_1) \in \mathbb{Z}.$$

Let x_0, x_1 be meridians around 0 and 1, lifted to the section of J. Since the section extends over the Mumford construction at ∞, we have the relation $x_0 x_1 = 1$. At a multiple fiber of order d, the two logarithmic fillings impose

$$x_0^3 = c^m, \qquad x_1^4 = c^n.$$

Proposition 4.1. *The fundamental group of $Y(u)$ is finite cyclic of order $|12\psi(a_0 + a_1)|$. In particular, under the hypotheses of Theorem 3.1, $Y(u)$ is simply connected.*

Proof. The preceding discussion gives

$$\pi_1(Y(u)) = \left\langle c, x_0, x_1 \;\middle|\; c \text{ central}, \quad x_0 x_1 = 1, \quad x_0^3 = c^m, \quad x_1^4 = c^n \right\rangle.$$

Eliminating $x_1 = x_0^{-1}$ makes the group abelian, generated by x_0 and c, and the remaining relations reduce this abelian group to a cyclic group of order $|4m + 3n|$. □

5. The integral homology

Let $f : Y(u) \to \mathbb{P}^1$ be the torus fibration and put $V = \mathrm{Hom}(\Lambda, \mathbb{Z})$. Let $(\alpha_1, \alpha_2, \alpha_3, \psi)$ be the basis of V dual to $\mathcal{B}_0$, and put

$$p = 12\psi(a_0 + a_1) = 4m + 3n.$$

Over U, the local system $R^k f_* \mathbb{Z}$ has fiber $\wedge^k V$. The fibers over 0 and 1 are multiple bielliptic surfaces, while the fiber at ∞ is the unimodular Mumford fiber of Section 1. The following lemma packages the integral information supplied by these three local models.[2]

[2]Caveat lector: All the material in this section following this line was generated by ChatGPT 5.6-Sol and has not been checked, beyond plausibility.

Lemma 5.1. *Suppose $|p| = 1$. Then there are integral classes*

$$q \in \wedge^2 V, \qquad r \in \wedge^3 V, \qquad \mathrm{vol} \in \wedge^4 V$$

such that

k	0	1	2	3	4
$H^0(\mathbb{P}^1, R^k f_*\mathbb{Z})$	$\mathbb{Z}$	$12\mathbb{Z}\psi$	$2\mathbb{Z}q$	$2\mathbb{Z}r$	$12\mathbb{Z}\mathrm{vol}.$

Moreover,

$$H^1(\mathbb{P}^1, R^1 f_*\mathbb{Z}) = H^1(\mathbb{P}^1, R^2 f_*\mathbb{Z}) = 0,$$

and each of

$$H^2(\mathbb{P}^1, R^j f_*\mathbb{Z}), \qquad 0 \le j \le 2,$$

is infinite cyclic.

Proof. We compute on a smooth fiber, then at the three special fibers, and finally on the base. On V, the contragredient monodromies are

$$\begin{aligned} T_0\alpha_1 &= -\alpha_1 - \alpha_2, & T_1\alpha_1 &= \alpha_2, \\ T_0\alpha_2 &= \alpha_1, & T_1\alpha_2 &= -\alpha_1 + 2\psi, \\ T_0\alpha_3 &= \alpha_2 + \alpha_3, & T_1\alpha_3 &= \alpha_1 - \psi + \alpha_3, \\ T_0\psi &= \psi, & T_1\psi &= \psi. \end{aligned}$$

Define the following vectors:

$$\begin{aligned} q &= -\alpha_1 \wedge \alpha_2 - 2\alpha_1 \wedge \psi - 4\alpha_2 \wedge \psi - 6\alpha_3 \wedge \psi, \\ r &= -\alpha_1 \wedge \alpha_2 \wedge \psi, & \mathrm{vol} &= \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \psi, \\ h_0 &= \alpha_1 + 2\alpha_2 + 3\alpha_3, & h_1 &= \alpha_1 + \alpha_2 + 2\alpha_3. \end{aligned}$$

Direct substitution gives

$$H^0(U, R^k f_*\mathbb{Z}) = \mathbb{Z},\ \mathbb{Z}\psi,\ \mathbb{Z}q,\ \mathbb{Z}r,\ \mathbb{Z}\mathrm{vol} \qquad (0 \le k \le 4),$$

and, for $i = 0, 1$,

$$V^{T_i} = \langle \psi, h_i \rangle, \qquad (\wedge^2 V)^{T_i} = \langle h_i \wedge \psi, q \rangle.$$

Let $d_0 = 3$, $d_1 = 4$, and let $\pi_i : F_i \to G_i$ be the unramified cover from the smooth good-reduction fiber to the reduced bielliptic fiber. For $0 \le k \le 4$, write

$$\mathrm{sp}_i^k := \pi_i^* : H^k(G_i, \mathbb{Z}) \longrightarrow H^k(F_i, \mathbb{Z})^{T_i} = (\wedge^k V)^{T_i}.$$

The affine group $\pi_1(G_i)$ is generated by the translations t_λ and by $\widetilde{\rho}_i$, with relations

$$\widetilde{\rho}_i t_\lambda \widetilde{\rho}_i^{-1} = t_{T_i\lambda}, \qquad \widetilde{\rho}_i^{d_i} = t_{v_i}.$$

Thus

$$\mathrm{im}(\mathrm{sp}_i^1) = \{\theta \in V^{T_i} : d_i \mid \theta(v_i)\}.$$

Since $|p| = 1$, $\psi(v_0) = m$ is a unit modulo 3 and $\psi(v_1) = n$ is a unit modulo 4. Evaluation on v_i therefore gives

$$V^{T_i}/\mathrm{im}(\mathrm{sp}_i^1) \simeq \mathbb{Z}/d_i\mathbb{Z},$$

generated by the class of ψ.

The remaining indices follow from the intersection pairing. In the bases $\langle h_i \wedge \psi, q\rangle$, its Gram matrices are, up to changing the sign of the first basis vector,

$$\begin{pmatrix} 0 & 3 \\ 3 & 12 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 2 & 12 \end{pmatrix}.$$

Indeed, abelianization gives

$$H_1(G_i, \mathbb{Z}) = (\Lambda_{T_i} \oplus \mathbb{Z}\widetilde{\rho}_i)/\langle (v_i, -d_i)\rangle,$$

where the matrices give $\Lambda_{T_i} \simeq \mathbb{Z}^2$. The last relation is primitive, so $H_1(G_i, \mathbb{Z})$ is free of rank 2. Poincaré duality and $e(G_i) = 0$ then imply that $H^*(G_i, \mathbb{Z})$ is torsion free and that its degree-2 pairing is unimodular. The cohomological transfer $(\pi_i)_*$ satisfies $(\pi_i)_*\pi_i^* = d_i$, so every sp_i^k is injective. Since pullback multiplies the degree-2 pairing by d_i, comparison of determinants gives

$$[(\wedge^2 V)^{T_0} : \mathrm{im}(\mathrm{sp}_0^2)] = 1, \qquad [(\wedge^2 V)^{T_1} : \mathrm{im}(\mathrm{sp}_1^2)] = 2.$$

The latter cokernel is generated by q: if q descended at the order-4 fiber, its square downstairs would be $q^2/4 = 3$, whereas the intersection form of G_1 is even (its canonical class is torsion and $H^2(G_1, \mathbb{Z})$ is torsion free).

Perfect duality gives the same two indices in degree 3. To identify the order-2 cokernel, replace h_1 by $h_1 - k\psi$ so that it descends. Its pairing with r is still 2, and hence is not divisible by 4; thus r does not descend. In degree 4 the indices are the covering degrees. Writing $[A : B]$ for the index of a finite-index subgroup $B \subset A$, we obtain

$$[(\wedge^k V)^{T_i} : \mathrm{im}(\mathrm{sp}_i^k)] = \begin{array}{c|ccccc} i\backslash k & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 3 & 1 & 1 & 3 \\ 1 & 1 & 4 & 2 & 2 & 4. \end{array}$$

At ∞, the toric cellular complex of the unimodular two-triangle model identifies the specialization map from the cohomology of the Mumford fiber with the full T_∞-invariant lattice. Thus no further index occurs, and intersecting the three specialization lattices gives the asserted row of global sections.

It remains to take cohomology on the base. For $j : U \hookrightarrow \mathbb{P}^1$ there is an exact sequence

$$0 \to R^k f_*\mathbb{Z} \to j_*(\wedge^k V) \to Q^k \to 0,$$

where

$$Q^1 = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \qquad Q^2 = Q^3 = \mathbb{Z}/2\mathbb{Z}, \qquad Q^4 = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Local-coefficient duality on the pair of pants gives $H^2(\mathbb{P}^1, j_*L) = L_{\langle T_0,T_1\rangle}$. Row reduction gives

$$V_{\langle T_0,T_1\rangle} = \mathbb{Z}[\alpha_3], \qquad (\wedge^2 V)_{\langle T_0,T_1\rangle} = \mathbb{Z}[\psi \wedge \alpha_3], \qquad [q] = 12[\psi \wedge \alpha_3].$$

Since Q^k has no H^2, this proves the assertion about H^2. Finally, the invariant lines surject onto Q^1 and Q^2. For $L = V, \wedge^2 V$, respectively, the required H^1 group is

$$\frac{(T_0 - I)L \cap (T_\infty - I)L}{(T_0 - I)L^{T_1}}.$$

For $L = V$ the numerator and denominator are $\mathbb{Z}\alpha_1$; for $L = \wedge^2 V$ they are $\mathbb{Z}(\alpha_1 \wedge \psi)$. Both groups therefore vanish. □

Choose the generators

$$\xi_0 = \omega, \qquad \xi_1 = \omega[\alpha_3], \qquad \xi_2 = \omega[\psi \wedge \alpha_3],$$

where ω is a generator of $H^2(\mathbb{P}^1, \mathbb{Z})$ and the common sign is fixed below.

Lemma 5.2. *The three transgressions relevant below are*

$$\begin{aligned} d_2(12\psi) &= p\xi_0, \\ d_2(2q) &= p\xi_1, \\ d_2(2r) &= p\xi_2. \end{aligned}$$

Proof. The first differential measures the incompatibility of local lifts of 12ψ. The relations

$$\widetilde{\rho}_0^3 = t_{v_0}, \qquad \widetilde{\rho}_1^4 = t_{v_1}$$

show that the two finite fibers contribute $12\psi(v_0)/3 = 4m$ and $12\psi(v_1)/4 = 3n$. The tautological cusp gluing contributes zero. Hence, after fixing the common orientation, $d_2(12\psi) = p\xi_0$.

Write

$$d_2(2q) = p'\xi_1, \qquad d_2(2r) = p''\xi_2.$$

The spectral sequence is multiplicative and d_2 is a derivation. The identities

$$\psi \wedge q = r, \qquad [q] = 12[\psi \wedge \alpha_3]$$

give $p'' = 2p - p'$. Applying d_2 to $(2q)(2r) = 0$ gives $|p'| = |p''|$, since both resulting pairings are ± 2. As $p \neq 0$, we cannot have $p'' = -p'$. Thus $p' = p'' = p$. □

Theorem 5.3. *If* $|12\psi(a_0 + a_1)| = 1$, *then*

$$H^k(Y(u), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, 6, \\ 0, & 1 \le k \le 5. \end{cases}$$

Thus $Y(u)$ *is an integral homology* 6*-sphere.*

Proof. The only differentials relevant in total degrees at most 3 are the three maps of Lemma 5.2. When $|p| = 1$, all three are isomorphisms. Lemma 5.1 then gives

$$H^1(Y(u), \mathbb{Z}) = H^2(Y(u), \mathbb{Z}) = H^3(Y(u), \mathbb{Z}) = 0.$$

The manifold $Y(u)$ is closed and oriented of real dimension 6. Poincaré duality gives the remaining assertion. $\square$

6. Additional remarks

Remark 6.1. Other than the discrete parameters in Theorem 3.1, there is the continuous parameter $u \in \Delta^*$. We may thus build a proper, holomorphic submersion $Y^* \to \Delta^*$ whose fiber over u is $Y(u)$, and similarly, for $X(u)$, a proper holomorphic submersion $X^* \to \Delta^*$ whose fiber over u is $X(u)$.

Does this family of complex manifolds degenerate in a nice way over the origin $0 \in \Delta$? The geometric construction is suggestive: The parameter u is a rescaling parameter for the linearization $\phi\colon t_{6P}^* M \to M$ discussed in Section 1. Thus as we take the limit $|u| \to 0$, the conformal ratio of the annular fundamental domain for multiplication of u acting on $\mathbb{G}_m$ grows to infinity. Since, furthermore, the linearization ϕ lifts the translation t_{6P}^{-1}, this strongly suggests the following: The limit

$$X(0) \coloneqq \lim_{u \to 0} X(u)$$

is the following object: Take the $\mathbb{P}^1$-bundle $\mathbb{P}_S(\mathcal{O} \oplus M)$ and glue the zero section to the infinity section by t_{6P}. This can thus be viewed fiberwise over $t \in \mathbb{P}^1$ as a "compactified semiabelian" degeneration of the corresponding family of complex 2-tori $X_t(u)$ as $u \to 0$.

Conjecture 6.2. *There is a flat, proper filling* $X \to \Delta$ *and the central fiber* $X(0)$ *is non-normal. The normalization is* $\mathbb{P}_S(\mathcal{O} \oplus M)$ *and the normalization map glues the zero and infinity sections via* t_{6P}.

For $Y^* \to \Delta^*$, which is particularly appealing, given that it is a smooth $\mathbb{S}^6$-bundle over the punctured disk, it is less clear what the limit should be, but it is natural to guess that it is a logarithmic type transformation of $X(0)$, whose fibers over 0 and 1 have multiplicity 3 and 4, and whose reductions are "semiabelian type" degenerations of bielliptic surfaces.

Remark 6.3. While Section 5 contains some rather difficult verifications, there are easier checks which can be used as positive indications that $Y(u) \simeq_{\mathrm{diffeo}} \mathbb{S}^6$. For example, the Mumford construction $Y(u)_\infty$ has exactly two 0-dimensional toric strata, and all other (reduced) fibers are either bielliptic or complex 2-tori. Hence we see immediately that $\chi_{\mathrm{top}}(Y(u)) = 2$ as required.

Remark 6.4. The rank 4 local system of first homologies $H_1(Y(u)_t, \mathbb{Z})$ contains a natural rank 3 sub-local system, given by the kernel of the unique invariant (up to scale) covector ψ of Lemma 2.13. Geometrically, this rank 3 sub-local system corresponds to a real 3-dimensional subtorus of the fibers. Quotienting fiberwise by the translational action of this subtorus, we reduce the fibers $Y(u)_t$ to circles, but some care must be taken at $t = 0, 1, \infty \in \mathbb{P}^1$.

At $t = \infty$, the kernel of ψ contains the vanishing cycles $W_{-2}H_1$ of the nearby fiber. The quotient by the vanishing cycles torus can be interpreted in the following manner: Take the Clemens collapse from the nearby fiber to $Y(u)_\infty$ and then compose with the moment mapping on the glued dP_6 to a glued hexagon (which is a polytope of this Mumford construction).

There is, in the construction, a distinguished edge of the glued hexagon. Indeed $\mathrm{Aut}^0(Y(u)) \simeq \mathbb{C}^*$ and the fixed locus of this $\mathbb{C}^*$-action is one of the three glued edges. This distinguished edge determines a further quotient of the hexagonal torus to a circle. Thus over ∞, this quotient map of the first paragraph extends continuously at ∞ to the composition of the moment mapping with a further quotient to a circle.

At 0 or 1, the situation is slightly different. Here, the rank 3 sub-local system contains the local rank 2 sub-local system on which the monodromy about 0 or 1 is nontrivial. Quotienting by the corresponding subtorus gives a multiple elliptic curve E'/μ_3 or E'/μ_4 because the twisted action defining the logarithmic transform producing the bielliptic multiple fiber acts by a torsion translation on E'. The further quotient collapses E'/μ_3 or E'/μ_4 to a circle, with multiplicity 3 or 4.

The conclusion is that the quotient by the torus associated to this natural rank 3 sub-local system is a map $\mathbb{S}^6 \to Q$ where Q is a Seifert fibered 3-manifold over $\mathbb{P}^1 \simeq \mathbb{S}^2$ with a multiple circle of multiplicity 3 over 0 and multiplicity 4 over 1. The condition $|12\psi(a_0 + a_1)| = 1$ in Theorem 3.1 is exactly translating to the condition $Q \simeq \mathbb{S}^3$ with $\mathbb{S}^3 \to \mathbb{S}^2$ the Seifert fibration with multiplicities 3 and 4 over two points. Thus the quotient map is a map $\mathbb{S}^6 \to \mathbb{S}^3$.